

新灯塔国际教育

AL 数学逆袭宝典

A-Level Pure Mathematics • Basis

Edexcel IAL P1 / P2

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写在前面

课堂上，常看到这样的情形：学生能听懂讲解，方法也清楚，但一到自己动笔，就总在计算上出错。分数运算、指数变形、因式分解、移项去分母——这些基础步骤本应扎实，却常常成为失分的主要原因。问题往往不在“会不会想”，而在计算基本功还不够稳。

这本《AL 数学逆袭宝典》正是为这类同学准备的。全书紧扣 Edexcel IAL P1 和 P2，把代数与函数中最重要的基础拆解开，一步步练扎实。每章安排了三个层次：入门练习，帮你建立手感；综合练习，巩固核心技能；还放了少量培优题，供有余力的同学进一步思考。目标很明确——把基础算对、算稳，为后面的学习铺好路。

基础暂时薄弱，不代表学不好 A Level。把计算基本功打扎实，后面的路会宽得多。

新灯塔国际教育

1 Number & Sequences (数与数列)

Fractions, negative numbers, exact values; arithmetic and simple geometric sequences.

Rules 规则

- Fractions (分数): $\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}$; $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$; $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$.
- Negatives (负号): $(-a)(-b) = ab$. When substituting a negative number, always enclose it in brackets.
- Exact value (精确值): Leave your answer as a fraction or surd unless a decimal is requested.
- Arithmetic sequence (等差数列): $u_n = a + (n-1)d$, $S_n = \frac{n}{2}(2a + (n-1)d) = \frac{n}{2}(a + \ell)$.
- Geometric sequence (等比数列): $u_n = ar^{n-1}$, $S_n = \frac{a(r^n - 1)}{r - 1}$ ($r \neq 1$).

Worked Examples 例题

Example 1. Calculate $\frac{3}{4} - \frac{5}{6}$.

$$\frac{3}{4} - \frac{5}{6} = \frac{9-10}{12} = -\frac{1}{12}.$$

Example 2. An arithmetic progression has first term $a = 5$ and common difference $d = 3$. Find the tenth term and the sum of the first ten terms.

$$u_{10} = 5 + 9 \cdot 3 = 32, \quad S_{10} = \frac{10}{2}(5 + 32) = 185.$$

Example 3. A geometric progression begins 2, 6, 18, ... Find the sixth term and the sum of the first six terms.

$$r = 3, \quad u_6 = 2 \cdot 3^5 = 486, \quad S_6 = \frac{2(3^6 - 1)}{3 - 1} = 728.$$

Example 4. Evaluate $\frac{(-2)^3 \cdot 3^{-1}}{4^{-1}}$.

$$= \frac{-8 \cdot \frac{1}{3}}{\frac{1}{4}} = \frac{-8}{3} \cdot 4 = -\frac{32}{3}.$$

Basic Drill (入门综合) · 50

1. Evaluate $\frac{2}{3} + \frac{1}{6}$.
2. Evaluate $\frac{5}{8} - \frac{1}{4}$.
3. Evaluate $\frac{3}{5} \times \frac{10}{9}$.
4. Evaluate $\frac{7}{8} \div \frac{14}{3}$.
5. Simplify $\frac{12}{18}$.
6. Compute $(-3) + 7$.
7. Compute $(-5) - (-2)$.
8. Compute $(-4) \times (-6)$.
9. Compute $18 \div (-3)$.
10. Evaluate $\frac{-2}{5} + \frac{3}{10}$.
11. Evaluate $1 - \frac{5}{7}$.
12. Evaluate $\left(-\frac{2}{3}\right)^2$.
13. Evaluate $\left(-\frac{3}{4}\right)^3$.
14. Find $\frac{1}{2}$ of 18.
15. Express 0.2 as a fraction in lowest terms.
16. Which is larger: $\frac{5}{8}$ or $\frac{3}{5}$?
17. Compute $\frac{2}{3} \times \frac{9}{4} \div \frac{1}{2}$.
18. Simplify $\frac{(-6) \times 4}{-8}$.
19. Evaluate $|-7| + |3|$.
20. Evaluate $|-5| - |-2|$.
21. AP: $a = 4, d = 2$. Find u_8 .
22. AP: $3, 7, 11, \dots$. Find u_{12} .
23. AP: $a = 10, d = -3$. Find u_7 .
24. AP: $u_1 = 2, u_2 = 9$. Find d and u_5 .
25. AP: $a = 1, d = 4$. Find S_6 .
26. AP: $5, 8, 11, \dots$. Find S_{10} .
27. AP: $a = 20, d = -2$. Find S_8 .
28. In an AP, $u_3 = 11, u_7 = 23$. Find a and d .
29. AP: $a = 6, \ell = u_n = 30, n = 9$. Find d .
30. AP: $S_5 = 40, a = 4$. Find d .
31. GP: $a = 3, r = 2$. Find u_6 .
32. GP: $2, 6, 18, \dots$. Find u_5 .
33. GP: $a = 81, r = \frac{1}{3}$. Find u_4 .

34. GP: $5, 10, 20, \dots$. Find S_5 .
35. GP: $a = 4, r = -2$. Find u_5 .
36. GP: $a = 1, r = 3$. Find S_4 .
37. Find the common ratio of $4, 12, 36, \dots$.
38. GP: $u_2 = 6, u_4 = 54, r > 0$. Find r and a .
39. AP or GP? $2, 5, 8, 11$.
40. AP or GP? $3, 6, 12, 24$.
41. Evaluate $\frac{3}{7} + \frac{2}{7}$.
42. Evaluate $2\frac{1}{3} - \frac{5}{6}$.
43. Write $\frac{15}{4}$ as a mixed number.
44. Write $3\frac{2}{5}$ as an improper fraction.
45. AP: first term -4 , common difference 5 . Find u_6 .
46. GP: first term -2 , common ratio -3 . Find u_4 .
47. Find S_{20} for AP $1, 2, 3, \dots, 20$.
48. If $u_n = 3n - 1$, find u_1 and u_{10} .
49. If $u_n = 2 \cdot 5^{n-1}$, find u_3 .
50. Simplify $\frac{(-1)^{10}}{4}$.

Practice (综合练习) · 25

Practice

- Evaluate $\frac{2}{3} - \frac{5}{6}$, divide the result by $\frac{1}{4}$, then add $(-2)^3$.
- AP: $u_3 = 14, u_8 = 29$. Find a, d, u_{15} , and S_{15} .
- A GP has $S_5 = 155$ and $r = 2$. Find a and u_6 .
- Insert three arithmetic means between 2 and 26 , and verify that the middle mean equals the average of the endpoints.
- A salary starts at \$24 000 and rises by \$1 200 each year. Find the smallest n such that total earnings over the first n years exceed \$300 000.
- An AP has $S_n = n(2n + 1)$. Find u_n, a , and d .

7. If $\frac{x}{3} = \frac{y}{5}$ and $\frac{y}{4} = \frac{z}{12}$, find $\frac{x}{z}$.
8. GP: $u_2 = 12$, $u_5 = 96$, $r > 0$. Find r , a , and S_4 .
9. Evaluate $\frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{9 \cdot 10}$ as a single fraction.
10. AP and GP both have $u_1 = 3$ and $u_4 = 24$. Find all possible d and r .
11. Solve $\frac{x-2}{5} + \frac{x+1}{3} = 2$, giving x as an exact fraction.
12. A ball drops from 12 m and rebounds to $\frac{2}{3}$ of the previous height each time. Find the total vertical distance traveled when it hits the ground for the 4th time.
13. If $S_n = 3n^2 - n$, find u_n and verify that $u_5 + u_6 = S_6 - S_4$.
14. GP: $S_3 = 26$, $S_6 = 728$, $r > 0$. Find a and r .
15. How many terms of the AP 7, 11, 15, ... must be summed to exceed 1000?
16. Evaluate $\frac{(-\frac{1}{2})^4 \cdot 2^5}{4^{-1}}$.
17. Three positive numbers in AP have sum 21 and product 315. Find the three numbers.
18. An infinite GP has first term 12 and sum 18. Find r and S_4 .
19. The AP with $u_n = 2n - 1$ has $S_n = n^2$. Show this by using $S_n - S_{n-1}$.
20. Find n if $2 + 5 + 8 + \cdots + (3n - 1) = 222$.
21. An AP has $a = -5$, $d = 4$; a GP has $a = 2$, $r = 3$. Find u_8 of the AP plus v_5 of the GP.
22. Evaluate $\frac{3/4}{2/5} - \frac{5/6}{1/3}$.
23. If $u_n = n^2 + n$, find n when $u_n = 132$.
24. A GP with positive terms has $u_2 + u_3 = 25$ and $u_3 + u_4 = 50$. Find a and r .

25. An AP has first term 8 and $S_{10} = 350$. Find d and check using $S_n = \frac{n}{2}(2a + (n-1)d)$.

Advanced Practice (培优) · 10

Advanced Practice

1. For an AP with an odd number of terms n , prove $S_n = n u_{(n+1)/2}$.
2. A GP with positive terms has $u_1 + u_2 = 15$ and $u_3 + u_4 = 60$. Find a and r .
3. In an AP, the 2nd, 5th and 11th terms form a GP. Express the GP ratio in terms of a and d .
4. Find all integers $n > 1$ such that $1 + 2 + \cdots + n$ is a perfect square.
5. A GP satisfies $S_2 = 6$ and $S_4 = 30$ with $r \neq 1$. Find a and r .
6. If a, b, c are in AP and b, c, d are in GP with $b, c \neq 0$, prove $ad = bc$.
7. Find $S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}}$ and $\lim_{n \rightarrow \infty} S_n$.
8. Three numbers in GP have sum 26 and product 216. Find them.
9. An AP satisfies $u_1 + \cdots + u_{2n} = 3(u_1 + \cdots + u_n)$ with $d \neq 0$. Find $\frac{u_{n+1}}{u_1}$.
10. An infinite GP has sum 4 and second term 1. Find a and r with $|r| < 1$.

2 Indices (指数)

Index laws for negative and fractional powers.

Rules 规则

- $a^m a^n = a^{m+n}$, $\frac{a^m}{a^n} = a^{m-n}$, $(a^m)^n = a^{mn}$.
- $(ab)^n = a^n b^n$, $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$.
- $a^0 = 1$ ($a \neq 0$), $a^{-n} = \frac{1}{a^n}$.
- $a^{1/n} = \sqrt[n]{a}$, $a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$.
- Rewrite $\frac{k}{x^n} = kx^{-n}$, $\sqrt{x} = x^{1/2}$.

Worked Examples 例题

Example 1. Simplify $2^3 \cdot 2^5$ and $\frac{x^7}{x^2}$.

$$2^8 = 256, \quad x^5.$$

Example 2. Simplify 5^{-2} and $(2x^{-3})^{-2}$.

$$\frac{1}{25}; \quad \frac{1}{4}x^6 = \frac{x^6}{4}.$$

Example 3. Write $\sqrt[3]{8x^6}$ as a power.

$$8^{1/3}x^{6/3} = 2x^2.$$

Example 4. Express $y = x^2 + \frac{3}{x} - 4\sqrt{x}$ using powers of x only.

$$y = x^2 + 3x^{-1} - 4x^{1/2}.$$

Basic Drill (入门综合) · 50

1. Simplify $3^2 \cdot 3^4$.
2. Simplify $5^7 \div 5^3$.
3. Simplify $(2^3)^2$.
4. Simplify $x^4 \cdot x^5$.
5. Simplify $\frac{a^9}{a^4}$.
6. Simplify $(y^3)^4$.
7. Simplify $2^3 \cdot 2$.
8. Simplify 7^0 .
9. Simplify x^0 ($x \neq 0$).
10. Simplify 4^{-1} .
11. Simplify 3^{-2} .
12. Simplify 10^{-3} .
13. Simplify $\left(\frac{1}{2}\right)^{-3}$.
14. Simplify x^{-4} .
15. Simplify $2x^{-1}$.
16. Write $\frac{1}{8}$ as a power of 2.
17. Write 9 as a power of 3.
18. Simplify $\sqrt{49}$.
19. Simplify $\sqrt[3]{27}$.
20. Simplify $16^{1/2}$.
21. Simplify $8^{1/3}$.
22. Simplify $27^{2/3}$.
23. Simplify $4^{3/2}$.
24. Simplify $9^{-1/2}$.
25. Simplify $(8x^3)^{1/3}$.
26. Simplify $(25y^4)^{1/2}$.
27. Simplify $x^{1/2} \cdot x^{3/2}$.
28. Simplify $\frac{x^{5/2}}{x^{1/2}}$.
29. Simplify $(x^{2/3})^3$.
30. Simplify $2^3 \cdot 5^3$ as a single power if possible, or evaluate.
31. Simplify $(2a^2b)^3$.
32. Simplify $\frac{6x^5}{2x^2}$.
33. Simplify $\frac{12a^3}{3a^{-2}}$.

34. Simplify $(3x^{-2})(4x^5)$.
35. Simplify $(2x^3)^{-2}$.
36. Write $\frac{5}{x^2}$ as kx^n .
37. Write $\frac{7}{x}$ as kx^n .
38. Write $4\sqrt{x}$ as kx^n .
39. Write $\frac{1}{\sqrt{x}}$ as kx^n .
40. Simplify $a^3 \div a^{-2}$.
41. Simplify $(a^{-1})^{-1}$.
42. Evaluate $32^{2/5}$.
43. Evaluate $81^{3/4}$.
44. Simplify $\left(\frac{x^3}{y^6}\right)^{1/3}$.
45. Simplify $(4^{-1})^{-2}$.
46. Simplify $x^2 \cdot x^{-2}$.
47. Simplify $5^2 \cdot 5^{-4}$.
48. Write 0.01 as a power of 10.
49. Simplify $(xy)^4$.
50. Simplify $\frac{(2x)^4}{4x^2}$.

Practice (综合练习) · 25

Practice

1. Simplify $\frac{(3x^{-2}y^3)^2 \cdot (2x^4y^{-1})}{6x^2y}$ completely.
2. Write $y = \frac{2}{x^2} + x\sqrt{x} - 5x^{-1}$ using powers of x only, then simplify $y \cdot x^2$.
3. Simplify $\frac{16^{3/4}}{2^{2/3} \cdot 8^{1/3}}$ to an integer.
4. Expand $(x^{1/2} - 2x^{-1/2})^2$ and write the result using positive indices only.
5. Given $2^{2x-1} = 8^{x+1}$, find x .
6. Simplify $\left(\frac{27a^{-9}}{8b^6}\right)^{2/3}$ for $a, b > 0$.
7. Express $\frac{5}{\sqrt[3]{x^2}}$ as kx^n and evaluate when $x = 8$.

8. If $a = 3^x$, express $3^{2x+1} + 3^{x-1}$ in terms of a .
9. Simplify $\frac{(x^m)^3 \cdot x^{1-m}}{x^{2m-1}}$ for $m \in \mathbb{R}$.
10. Solve $5^{x+1} = 125^{x-1}$ by rewriting both sides as powers of 5.
11. Simplify $(2^{n+3})(2^{n-1}) \div 4^{n+1}$.
12. Evaluate $\left(\frac{8}{27}\right)^{-2/3} + \left(\frac{9}{4}\right)^{1/2}$.
13. Rewrite $y = \frac{x^2 + 3}{\sqrt{x}}$ as $Ax^p + Bx^q$ and find y when $x = 4$.
14. Show that $\frac{a^{-1}b^{-1}}{a^{-1} + b^{-1}} = \frac{1}{a + b}$.
15. Simplify $\left(\frac{x^{-3}y^2}{x^2y^{-4}}\right)^{-1/2}$ for $x, y > 0$.
16. Find n if $2^{n+1} \cdot 4^n = 128$.
17. Simplify $4^x \cdot 2^{3-2x} \cdot 8^{x-1}$ as a single power of 2.
18. If $f(x) = x^{-3/2}$, find $\frac{f(4x)}{f(x)}$ simplified.
19. Simplify $(8x^6y^{-9})^{2/3}$ for $x, y > 0$.
20. Given $2^x = 5$, express $\frac{4^{x+1}}{25}$ as an integer.
21. Simplify $\frac{(a^{1/2} + b^{1/2})^2 - (a^{1/2} - b^{1/2})^2}{4ab^{1/2}}$.
22. Solve $2^{x+1} + 2^{x-1} = 5$ by setting $u = 2^x$.
23. Write $\sqrt[4]{\frac{16x^8}{y^4}}$ in the form $\frac{ax^b}{cy^d}$ for $x, y > 0$.
24. Simplify $9^x \cdot 3^{1-2x}$ as a single power of 3.

25. If $2^x = 3$ and $3^y = 2$, express 6^{xy} as an integer.

Advanced Practice (培优) · 10

Advanced Practice

1. Solve $4^x - 2^{x+1} - 8 = 0$ by setting $u = 2^x$.
2. Simplify $\frac{(a^{1/2} + b^{1/2})(a^{1/2} - b^{1/2})}{a - b}$ for $a \neq b$.
3. Given $x = 3^t + 3^{-t}$, show that $x^2 = 3^{2t} + 3^{-2t} + 2$, and hence express $x^2 - 2$ as a sum of powers of 3.
4. Find all real $x > 0$ such that $(x^2)^x = x^4$.
5. Write $\sqrt[4]{\frac{16x^8}{81y^4}}$ in the form $\frac{ax^b}{cy^d}$.
6. Prove $\left(\frac{a^m}{a^n}\right)^p = \frac{a^{mp}}{a^{np}}$ using index laws only.
7. If $2^x = 5^y = 10^z$, show $\frac{1}{z} = \frac{1}{x} + \frac{1}{y}$.
8. Simplify $\left(\frac{x^{a+b}}{x^{a-b}}\right)^{a+b} \cdot \left(\frac{x^b}{x^a}\right)^{a^2-b^2}$ to a single power of x .
9. For $y = \frac{x^2 + 1}{\sqrt{x}}$, rewrite as a sum of powers $Ax^p + Bx^q$.
10. Solve $x^{x^2-3x+2} = 1$ for $x > 0$.

3 Surds (根式)

Simplify surds, expand surd expressions, and rationalise denominators.

Rules 规则

- For $a, b \geq 0, b \neq 0$: $\sqrt{a}\sqrt{b} = \sqrt{ab}$ and $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$.
- To simplify a surd, look for square factors, e.g. $\sqrt{18} = \sqrt{9 \cdot 2} = 3\sqrt{2}$.
- Rationalising (有理化): $\frac{1}{\sqrt{a}} = \frac{\sqrt{a}}{a}$; $\frac{1}{a + \sqrt{b}} = \frac{a - \sqrt{b}}{a^2 - b}$.
- $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b$.

Worked Examples 例题

Example 1. Simplify $\sqrt{50} + \sqrt{8}$.

$$5\sqrt{2} + 2\sqrt{2} = 7\sqrt{2}.$$

Example 2. Rationalise $\frac{6}{\sqrt{3}}$.

$$\frac{6\sqrt{3}}{3} = 2\sqrt{3}.$$

Example 3. Rationalise $\frac{1}{3 - \sqrt{2}}$.

$$\frac{3 + \sqrt{2}}{9 - 2} = \frac{3 + \sqrt{2}}{7}.$$

Example 4. Expand $(\sqrt{5} + 2)^2$.

$$5 + 4\sqrt{5} + 4 = 9 + 4\sqrt{5}.$$

Basic Drill (入门综合) · 50

1. Simplify $\sqrt{12}$.
2. Simplify $\sqrt{18}$.
3. Simplify $\sqrt{32}$.
4. Simplify $\sqrt{75}$.
5. Simplify $\sqrt{8}$.
6. Simplify $\sqrt{27}$.
7. Simplify $\sqrt{45}$.
8. Simplify $\sqrt{98}$.
9. Simplify $\sqrt{200}$.
10. Simplify $\sqrt{72}$.
11. Simplify $2\sqrt{3} + 5\sqrt{3}$.
12. Simplify $7\sqrt{2} - 3\sqrt{2}$.
13. Simplify $\sqrt{12} + \sqrt{3}$.
14. Simplify $\sqrt{50} - \sqrt{8}$.
15. Simplify $\sqrt{18} + \sqrt{32}$.
16. Simplify $3\sqrt{8} - 2\sqrt{2}$.
17. Evaluate $\sqrt{9} \cdot \sqrt{4}$.
18. Simplify $\sqrt{6} \cdot \sqrt{6}$.
19. Simplify $\sqrt{2} \cdot \sqrt{8}$.
20. Simplify $\sqrt{3} \cdot \sqrt{12}$.
21. Simplify $\frac{\sqrt{18}}{\sqrt{2}}$.
22. Simplify $\frac{\sqrt{50}}{\sqrt{2}}$.
23. Rationalise $\frac{1}{\sqrt{2}}$.
24. Rationalise $\frac{1}{\sqrt{5}}$.
25. Rationalise $\frac{3}{\sqrt{3}}$.
26. Rationalise $\frac{5}{\sqrt{10}}$.
27. Rationalise $\frac{2}{\sqrt{8}}$.
28. Expand $(\sqrt{2} + 1)(\sqrt{2} - 1)$.
29. Expand $(\sqrt{3} + 2)(\sqrt{3} - 2)$.
30. Expand $(\sqrt{5} + \sqrt{2})(\sqrt{5} - \sqrt{2})$.
31. Expand $(1 + \sqrt{2})^2$.
32. Expand $(3 - \sqrt{2})^2$.
33. Expand $(\sqrt{3} + \sqrt{2})^2$.

34. Simplify $\sqrt{\frac{9}{16}}$.
35. Simplify $\sqrt{\frac{4}{25}}$.
36. Simplify $\sqrt{\frac{18}{2}}$.
37. Write $\sqrt{0.49}$ as a decimal.
38. Simplify $5\sqrt{7} - 2\sqrt{7} + \sqrt{7}$.
39. Simplify $\sqrt{a^2}$ for $a \geq 0$.
40. Simplify $\sqrt{4x^2}$ for $x \geq 0$.
41. Simplify $\sqrt{9x^4}$ for $x \geq 0$.
42. Rationalise $\frac{1}{2\sqrt{3}}$.
43. Simplify $\sqrt{28} + \sqrt{63}$.
44. Simplify $2\sqrt{12} + 3\sqrt{48}$.
45. Evaluate $(\sqrt{7})^2$.
46. Simplify $\frac{\sqrt{12}}{2}$.
47. Simplify $\sqrt{2} + \sqrt{2} + \sqrt{2}$.
48. Is $\sqrt{2} + \sqrt{8}$ equal to $\sqrt{10}$? (Yes/No).
49. Simplify $\sqrt{16 \cdot 3}$.
50. Write $4\sqrt{3}$ as a single square root if possible: $\sqrt{?}$.

Practice (综合练习) · 25

Practice

- Simplify $\sqrt{75} + \sqrt{48} - \sqrt{27} + \sqrt{12}$ and express the answer as $k\sqrt{3}$.
- Rationalise $\frac{5}{\sqrt{7} - \sqrt{2}}$ and write the result as $p\sqrt{7} + q\sqrt{2}$.
- Expand $(2\sqrt{3} - 1)^2$, then rationalise $\frac{1}{2\sqrt{3} - 1}$.
- Simplify $\frac{\sqrt{50} + \sqrt{18}}{\sqrt{2}} - \frac{\sqrt{32}}{\sqrt{2}}$.
- Given $x = \sqrt{5} + 2$, find $x^2 - 4x$ without using decimals.
- Rationalise $\frac{\sqrt{5} + 1}{\sqrt{5} - 1} + \frac{\sqrt{5} - 1}{\sqrt{5} + 1}$ and simplify to an integer.

7. Simplify $\sqrt{4a^3b^2}$ for $a, b \geq 0$, then evaluate when $a = 8$, $b = 3$.
8. Solve $\sqrt{x+8} = x+2$ and reject any extraneous root.
9. Show that $\frac{1}{\sqrt{n+1} + \sqrt{n}} = \sqrt{n+1} - \sqrt{n}$, then evaluate $\sum_{n=1}^9 \frac{1}{\sqrt{n+1} + \sqrt{n}}$.
10. Express $\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}$ with a rationalised denominator when $a = 12$, $b = 27$.
11. If $(\sqrt{a} + \sqrt{b})^2 = 23 + 4\sqrt{30}$, find $a, b > 0$.
12. Simplify $\sqrt{50x^2} - \sqrt{18x^2} + \sqrt{8x^2}$ for $x \geq 0$.
13. Rationalise $\frac{2 + \sqrt{3}}{2 - \sqrt{3}}$ and write the answer as $p + q\sqrt{3}$.
14. Compare $\sqrt{50} - \sqrt{32}$ and $2\sqrt{2}$ by simplifying each side.
15. Write $\frac{6}{\sqrt{12}} + \frac{3}{\sqrt{27}}$ as a single simplified surd.
16. Given $x = \sqrt{3} - 1$, find $x^2 + 2x + 1$.
17. Simplify $\sqrt{0.18} + \sqrt{0.32}$ as $k\sqrt{2}$ with rational k .
18. Expand and simplify $(\sqrt{5} - \sqrt{2})(\sqrt{5} + \sqrt{2})(\sqrt{5} + 1)$.
19. Solve $\sqrt{x+3} = 2\sqrt{x-1}$ for $x \geq 1$.
20. If $\sqrt{8} + \sqrt{18} = k\sqrt{2}$, find k , then simplify $\sqrt{k^2 + 1}$.
21. Rationalise $\frac{3\sqrt{2}}{\sqrt{6} + \sqrt{3}}$ and write the answer in simplest surd form.
22. Simplify $\sqrt{27} + \sqrt{12} - 2\sqrt{3}$.
23. Given $(\sqrt{a} - 1)^2 = 6 - 2\sqrt{5}$, find a .
24. Express $\sqrt{7 + 4\sqrt{3}}$ in the form $p + \sqrt{q}$ with integers p, q .

25. A rectangle has sides $(\sqrt{5} + 1)$ cm and $(\sqrt{5} - 1)$ cm. Find its area and perimeter in simplest form.

Advanced Practice (培优) · 10**Advanced Practice**

1. Rationalise $\frac{\sqrt{a} - \sqrt{b}}{\sqrt{a} + \sqrt{b}}$ and simplify.
2. Prove $\sqrt{4 + 2\sqrt{3}} = \sqrt{3} + 1$ by squaring the right-hand side.
3. Show $\sqrt{7 + 4\sqrt{3}} = 2 + \sqrt{3}$.
4. Find integers $m, n > 0$ such that $(\sqrt{m} + \sqrt{n})^2 = 11 + 2\sqrt{30}$.
5. Show $\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$.
6. Solve $\sqrt{x} + \sqrt{x-4} = 2$ for $x \geq 4$.
7. If $x = \sqrt{2} + \sqrt{8} + \sqrt{18}$, write $x = k\sqrt{2}$ and find k .
8. Expand and simplify $(\sqrt{5} + 2\sqrt{3})(\sqrt{5} - 2\sqrt{3} + 1)$.
9. Rationalise $\frac{\sqrt{6}}{\sqrt{2} + \sqrt{3}}$ and write the answer as $p\sqrt{2} + q\sqrt{3}$.
10. Simplify $\sqrt{a^2 + 2ab + b^2}$ and state when it equals $a + b$.

4 Expanding (展开)

Expanding single brackets, the FOIL method, perfect squares, and the difference of squares.

Rules 规则

- $a(b + c) = ab + ac.$
- $(x + a)(x + b) = x^2 + (a + b)x + ab.$
- $(x + a)^2 = x^2 + 2ax + a^2, \quad (x - a)^2 = x^2 - 2ax + a^2.$
- $(x + a)(x - a) = x^2 - a^2.$
- $(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd.$

Worked Examples 例题

Example 1: $3(2x - 5) = 6x - 15.$

Example 2: $(x + 4)(x + 7) = x^2 + 11x + 28.$

Example 3: $(2x - 3)(x + 5) = 2x^2 + 10x - 3x - 15 = 2x^2 + 7x - 15.$

Example 4: $(x + 2)^2 = x^2 + 4x + 4$ and $(3x - 1)^2 = 9x^2 - 6x + 1.$

Basic Drill (入门综合) · 50

1. Expand $4(x + 3).$
2. Expand $5(2x - 1).$
3. Expand $-3(x - 4).$
4. Expand $x(x + 5).$
5. Expand $2x(3x - 7).$
6. Expand $(x + 2)(x + 3).$
7. Expand $(x + 5)(x + 1).$
8. Expand $(x - 3)(x + 4).$
9. Expand $(x - 2)(x - 6).$
10. Expand $(x + 7)(x - 7).$
11. Expand $(x + 3)^2.$
12. Expand $(x - 5)^2.$

13. Expand $(x + 1)^2$.
14. Expand $(x - 1)^2$.
15. Expand $(2x + 1)(x + 3)$.
16. Expand $(2x - 5)(x + 2)$.
17. Expand $(3x + 2)(x - 4)$.
18. Expand $(3x - 1)(2x + 5)$.
19. Expand $(2x + 3)^2$.
20. Expand $(3x - 2)^2$.
21. Expand $(x + y)(x + 2y)$.
22. Expand $(a + b)(a - b)$.
23. Expand $(2a + 3b)(2a - 3b)$.
24. Expand $x(x^2 + 2x + 1)$.
25. Expand $3(x^2 - 4x + 5)$.
26. Expand $(x + 2)(x^2 + 3)$.
27. Expand $(x - 1)(x^2 + x + 1)$.
28. Expand $(x + 1)(x^2 - x + 1)$.
29. Expand $-2(3x - 5)$.
30. Expand $(5 - x)(5 + x)$.
31. Expand $(y + 4)(y + 4)$.
32. Expand $(2 + x)(3 + x)$.
33. Expand $(7 - 2x)(1 + x)$.
34. Expand $(4x + 1)(4x - 1)$.
35. Expand $(x + 0.5)(x + 2)$ (exact fractions ok).
36. Expand $6\left(x - \frac{1}{3}\right)$.
37. Expand $(x + 3)(2x + 3)$.
38. Expand $(5x - 2)(x - 3)$.
39. Expand $(x + 2y)^2$.
40. Expand $(3x - y)^2$.
41. Expand $(x - 2)(x + 2)(x + 1)$ step by step.
42. Expand $2(x + 1)(x + 3)$.
43. Expand $-(x - 4)(x + 1)$.
44. Expand $(x^2 + 2)(x^2 - 2)$.
45. Expand $(2x + 5)(3x + 1)$.
46. Collect: expand then simplify $2(x + 1) + 3(x - 2)$.
47. Expand $(x + 1)^3$ using $(x + 1)(x + 1)^2$.

48. Expand $(1 - x)^2$.

50. Expand $(x + 3)(x - 3) + (x + 1)^2$.

49. Expand $(2 - 3x)^2$.

Practice (综合练习) · 25**Practice**

1. Expand and simplify $(2x - 3)(x + 4) - (x - 1)(x + 2)$.
2. Expand $(x + 2)^3$, then evaluate the result at $x = -1$.
3. Expand $(2x - 1)^3$ and collect like terms.
4. Expand $(x + y + 1)(x + y - 1)$ and rewrite the answer as a difference of squares in $(x + y)$.
5. Expand $(3x + 2y)(x - 5y)$ and state the coefficient of xy .
6. Expand $(x^2 + 3x - 1)(x + 2)$ and find the coefficient of x^2 .
7. Expand $(2x^2 - x + 3)(x - 4)$ completely.
8. Expand $(n + 1)^2 - n^2$ and explain why the result is odd for every integer n .
9. Expand $\left(x + \frac{1}{x}\right)^2$ and $\left(x - \frac{2}{x}\right)^2$, then subtract the second from the first.
10. Expand $(1 + 2x)(1 - 2x + 4x^2)$ and simplify.
11. Expand and simplify $(x + 3)^2 - (x - 3)^2$, then factor the result.
12. Expand $(a + b)^2 - (a - b)^2$ and write the answer as a product.
13. Expand $(2x + 3)^2 - (x - 1)^2$ completely.
14. Find the coefficient of x in $(2x + 5)(3x - 7)$.
15. Expand $(x - 4)(x + 9)$ and state the constant term.

16. Expand $(x+2)(x+3)(x+4)$ and find the sum of the coefficients.
17. Expand $(1-x)(1+x+x^2+x^3)$ and simplify.
18. Expand $x(2x-1)(x+5)$ as a cubic polynomial.
19. Expand $(3-x)^2(x+1)$ completely.
20. Expand $5(2x-1) - 2(3x+4)$ and solve the resulting expression $= 0$.
21. Expand $(ax+b)(cx+d)$; for $a=c=2$, $b=1$, $d=-3$, write the quadratic obtained.
22. Expand $\left(2x + \frac{1}{2}\right)^2$ using exact fractions.
23. Expand $(x^2+1)^2$ and $(x-1)^2(x+1)^2$; show that they differ by $4x^2$.
24. Expand $(x+1)^2(x-1)^2$ as $(x^2-1)^2$ and verify by direct expansion.
25. Expand $(x-2)(x+2)(x^2+4)$ and simplify fully.

Advanced Practice (培优) · 10**Advanced Practice**

1. Expand $(x+a)(x+b)(x+c)$ and write each coefficient in terms of a , b and c .
2. Prove $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$ by direct expansion.
3. Find p if the coefficient of x in $(2x+p)(x-3)$ equals 7.
4. Expand $(1+x)^4$.
5. Show by expanding that $(x^2+xy+y^2)(x-y) = x^3-y^3$.
6. Expand $\left(x + \frac{1}{x} + 1\right)\left(x + \frac{1}{x} - 1\right)$ and simplify.
7. Find the coefficient of x^2 in $(2+3x)^5$.

8. Expand $(x+2)^2(x-2)^2$ and write the result as a square.
9. Find k such that $(x+k)^2 = x^2 + 6x + 9$.
10. Find the coefficient of x^2 in $(1+2x+x^2)(3-x+4x^2)$.

5 Factorising (因式分解)

Common factors, quadratic trinomials, difference of squares, and grouping.

Rules 规则

- Take out the common factor (公因式): $ax + ay = a(x + y)$.
- Factorise a quadratic trinomial: $x^2 + (a + b)x + ab = (x + a)(x + b)$.
- Apply the difference of squares: $x^2 - a^2 = (x - a)(x + a)$.
- Recognise a perfect square: $x^2 + 2ax + a^2 = (x + a)^2$.
- Factor by grouping (分组分解): $ax + ay + bx + by = a(x + y) + b(x + y) = (a + b)(x + y)$.

Worked Examples 例题

Example 1. Factorise $6x + 15$.

$$3(2x + 5).$$

Example 2. Factorise $x^2 + 5x + 6$.

$$(x + 2)(x + 3).$$

Example 3. Factorise $x^2 - 9$ and $4x^2 - 25$.

$$(x - 3)(x + 3); \quad (2x - 5)(2x + 5).$$

Example 4. Factorise $2x^2 + 7x + 3$.

$$(2x + 1)(x + 3).$$

Basic Drill (入门综合) · 50

1. Factorise $4x + 8$.
2. Factorise $9x - 12$.
3. Factorise $x^2 + 3x$.
4. Factorise $5x^2 - 10x$.
5. Factorise $6x^2 + 9x$.
6. Factorise $x^2 + 7x + 12$.
7. Factorise $x^2 + 9x + 20$.
8. Factorise $x^2 + x - 6$.
9. Factorise $x^2 - x - 12$.
10. Factorise $x^2 - 5x + 6$.
11. Factorise $x^2 - 8x + 16$.
12. Factorise $x^2 + 6x + 9$.
13. Factorise $x^2 - 16$.
14. Factorise $x^2 - 49$.
15. Factorise $9x^2 - 4$.
16. Factorise $25x^2 - 1$.
17. Factorise $2x^2 + 5x + 2$.
18. Factorise $3x^2 + 10x + 3$.
19. Factorise $2x^2 - x - 3$.
20. Factorise $3x^2 - 5x - 2$.
21. Factorise $x^2 - 2x$.
22. Factorise $14x - 21$.
23. Factorise $x^2 - 1$.
24. Factorise $x^2 + 4x + 4$.
25. Factorise $x^2 - 10x + 25$.
26. Factorise $x^2 + 11x + 24$.
27. Factorise $x^2 - 7x + 10$.
28. Factorise $x^2 + 2x - 15$.
29. Factorise $x^2 - 3x - 28$.
30. Factorise $4x^2 - 9$.
31. Factorise $16x^2 - 25$.
32. Factorise $2x^2 + 8x$.
33. Factorise $3x^2 - 12x$.
34. Factorise $x^2 + 8x + 15$.

35. Factorise $x^2 - 9x + 18$.

43. Factorise $x^2y + xy^2$.

36. Factorise $5x^2 + 5x$.

44. Factorise $x^2 - x - 2$.

37. Factorise $x^2 - 36$.

45. Factorise $x^2 + 5x - 14$.

38. Factorise $2x^2 + 3x + 1$.

46. Factorise $2x^2 - 8$.

39. Factorise $4x^2 + 4x + 1$.

47. Factorise $3x^2 - 27$.

40. Factorise $6x^2 + x - 2$.

48. Factorise $x^2 + 12x + 36$.

41. Factorise $ax + bx$.

49. Factorise $x^2 - 14x + 49$.

42. Factorise $x^3 + x^2$.

50. Factorise $10x^2 - 15x$.

Practice (综合练习) · 25**Practice**

1. Factorise completely $2x^4 - 18x^2$ and hence solve $2x^4 - 18x^2 = 0$.
2. Expand $(a + b)^2 - (a - b)^2$, then factorise your result.
3. Factorise $6x^2y - 15xy + 9y$ completely.
4. Factorise $x^2 - (y + 1)x + y$ and state for which y both factors are linear in x .
5. Factorise by grouping: $xy + 3y + 2x + 6$.
6. Factorise by grouping: $x^3 + 2x^2 + x + 2$.
7. Factorise $x^3 - 8$ and solve $x^3 = 8$.
8. Factorise $x^3 + 27$ and solve $x^3 = -27$.
9. Factorise $12x^2 - 5x - 2$ and solve $12x^2 - 5x - 2 = 0$.

10. Factorise $8x^2 + 22x + 15$.
11. Factorise $x^4 - 16$ completely over the reals and find all real x with $x^4 = 16$.
12. Factorise $x^4 + 4x^2 + 3$ and solve $x^4 + 4x^2 + 3 = 0$ over the reals (if any).
13. Factorise $2x^3 - 50x$ and solve $2x^3 - 50x = 0$.
14. Factorise $x^2 + 2xy + y^2 - 9$ as a difference of squares.
15. Factorise $x^2 - xy - 2y^2$ and $2x^2 + xy - y^2$.
16. Factorise $15x^2 + 14x - 8$.
17. Factorise $x^3 - x$ completely and list the three real roots of $x^3 - x = 0$.
18. Factorise $9x^2 - 30x + 25 - 4y^2$ as a difference of squares.
19. Factorise $4x^2 - 12xy + 9y^2$ and state the linear relation between x and y when it equals zero.
20. Factorise $6x^2 - x - 12$.
21. Given $(x + m)(x + n) = x^2 - 5x - 24$ with integers $m > n$, find m and n .
22. Factorise $x^4 - 5x^2 + 4$ completely over the reals.
23. Factorise $x^2 + 4x + 4 - y^2$.
24. For integer k , factorise $x^2 + kx + k$ when possible with integer factors; find one such $k \neq 0$.
25. Factorise $a^2 - 2ab + b^2 - c^2$ completely.

Advanced Practice (培优) · 10

Advanced Practice

1. Factorise $x^4 + 4$ by writing $x^4 + 4x^2 + 4 - 4x^2$.
2. Prove $a^3 + b^3 + c^3 - 3abc = \frac{1}{2}(a + b + c)[(a - b)^2 + (b - c)^2 + (c - a)^2]$. Deduce the case $a + b + c = 0$.
3. Find all integers k for which $x^2 + kx + 12$ factorises over the integers.
4. Factorise $x^6 - 1$ completely into real polynomials.
5. By factorising, show $n^3 - n$ is divisible by 6 for every integer n .
6. Factorise $x^2 + 4x + 1 - y^2$.
7. Factorise $2x^2 + (a + 4)x + 2a$ in terms of a .
8. Factorise $x^3 + 3x^2 - x - 3$ completely over the reals.
9. Find all monic linear factors over $\mathbb{Z}$ of $x^3 - 6x^2 + 11x - 6$.
10. Find all integers $n > 1$ such that $n^2 + n + 1$ divides $n^3 - 1$, by factorising $n^3 - 1$.

6 Algebraic Fractions (代数分式)

Simplify, add, subtract, multiply, and divide algebraic fractions.

Rules 规则

- Factorise the numerator and denominator, then cancel any common factors (约分).
- $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$, $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$.
- $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$.
- State any excluded values (values that make the denominator zero).

Worked Examples 例题

Example 1. Simplify $\frac{x^2 - 9}{x^2 + 5x + 6}$.

$$\frac{(x-3)(x+3)}{(x+2)(x+3)} = \frac{x-3}{x+2} \quad (x \neq -3, -2).$$

Example 2. $\frac{2}{x} + \frac{3}{x+1} = \frac{2(x+1) + 3x}{x(x+1)} = \frac{5x+2}{x(x+1)}$.

Example 3. $\frac{x^2}{x+2} \div \frac{x}{x^2-4} = \frac{x^2}{x+2} \cdot \frac{(x-2)(x+2)}{x} = x(x-2)$.

Basic Drill (入门综合) · 50

1. Simplify $\frac{6x}{9}$.
2. Simplify $\frac{4x^2}{8x}$.
3. Simplify $\frac{x^2}{x}$.
4. Simplify $\frac{5(x+1)}{10}$.
5. Simplify $\frac{3x+6}{3}$.
6. Simplify $\frac{x^2-1}{x-1}$ ($x \neq 1$).
7. Simplify $\frac{x^2-4}{x+2}$.
8. Simplify $\frac{x^2+5x}{x}$.
9. Simplify $\frac{2x^2+4x}{2x}$.
10. Simplify $\frac{x^2+3x+2}{x+1}$.
11. Simplify $\frac{x^2+7x+12}{x+3}$.
12. Simplify $\frac{x^2-9}{x^2-x-6}$.
13. Compute $\frac{1}{x} + \frac{1}{x}$.
14. Compute $\frac{2}{x} + \frac{3}{x}$.
15. Compute $\frac{1}{x} + \frac{1}{2}$.
16. Compute $\frac{1}{x} + \frac{1}{x+1}$.
17. Compute $\frac{3}{x} - \frac{1}{x}$.
18. Compute $\frac{5}{x} - \frac{2}{x+2}$.
19. Compute $\frac{x}{3} \times \frac{6}{x^2}$.
20. Compute $\frac{2x}{5} \times \frac{10}{4x}$.
21. Compute $\frac{x^2}{3} \div \frac{x}{6}$.
22. Compute $\frac{4}{x} \div \frac{2}{x^2}$.
23. Simplify $\frac{(x+1)^2}{x+1}$.
24. Simplify $\frac{x^2-16}{x^2-8x+16}$.
25. Simplify $\frac{2x+4}{x^2+2x}$.
26. Simplify $\frac{6x^2y}{3xy}$.
27. Simplify $\frac{x^3}{x^2}$.
28. $\frac{1}{2x} + \frac{1}{4x}$.
29. $\frac{3}{2x} - \frac{1}{4x}$.
30. Simplify $\frac{x^2-x}{x^2-1}$.

31. Simplify $\frac{x^2 + 4x + 3}{x^2 + 5x + 6}$.

32. $\frac{x}{x+1} \times \frac{x+1}{x^2}$.

33. $\frac{x+2}{x} \div \frac{x+2}{3}$.

34. Simplify $\frac{5x-10}{x-2}$.

35. Simplify $\frac{x^2-2x}{x^2-4}$.

36. $\frac{2}{x+3} + \frac{1}{x+3}$.

37. $\frac{4}{x-1} - \frac{2}{x-1}$.

38. Simplify $\frac{9x^2}{12x}$.

39. Simplify $\frac{(2x)^2}{4x}$.

40. $\frac{1}{x^2} \times x^3$.

41. Simplify $\frac{x^2+2x+1}{x+1}$.

42. Simplify $\frac{x^2-6x+9}{x-3}$.

43. $\frac{3x}{4} \times \frac{8}{9x}$.

44. $\frac{x}{5} \div \frac{x}{10}$.

45. Simplify $\frac{2x^2-8}{2x+4}$.

46. State excluded value for $\frac{1}{x-5}$.

47. State excluded values for $\frac{x}{x^2-9}$.

48. $\frac{1}{x} + \frac{2}{3x}$.

49. Simplify $\frac{x^2-25}{5-x}$.

50. $\frac{x+1}{2} \div \frac{x+1}{4}$.

Practice (综合练习) · 25**Practice**

1. Simplify $\frac{x^2-5x+6}{x^2-9} \cdot \frac{x+3}{x-2}$.

2. Simplify $\frac{x^2-4}{x^2+x-6} \div \frac{x-2}{x+3}$.

3. $\frac{2}{x-1} - \frac{3}{x+1}$.

4. $\frac{x}{x+2} + \frac{3}{x-2}$.

5. $\frac{1}{x^2-1} + \frac{1}{x-1}$.

6. Simplify $\frac{2x^2+x-3}{x^2-1}$.

7. $\frac{4}{x^2-4} - \frac{1}{x-2}$.

8. Simplify $\frac{x^3-8}{x^2-4}$ for $x \neq \pm 2$ (use $x^3-8 = (x-2)(\dots)$).

9. $\frac{3x}{x+4} - \frac{2}{x}$ single fraction.

10. Simplify $\left(\frac{1}{a} - \frac{1}{b}\right) \div \left(\frac{1}{a^2} - \frac{1}{b^2}\right)$ for $a \neq \pm b$.

11. Simplify $\frac{1}{x+h} - \frac{1}{x}$ as a single fraction in x and h ($x \neq 0, x+h \neq 0$).

12. $\frac{x^2}{x^2-1} - \frac{x}{x-1}$.

13. Simplify $\frac{6x^2-x-2}{3x^2+5x-2}$.

14. $\frac{2x+1}{x^2+3x} + \frac{1}{x}$.

15. Simplify $\frac{x}{x-1} - \frac{x}{x+1}$ and state excluded values.

16. $\frac{x-1}{x+1} \div \frac{x^2-1}{x}$.

17. $\frac{5}{x^2+x} - \frac{3}{x+1}$.

18. Solve $\frac{x}{x-1} + \frac{1}{x+1} = 2$ (state exclusions).

19. $\frac{x}{2x-1} - \frac{x+1}{2x+1}$.

20. Write $\frac{3}{x} - \frac{2}{x^2} + \frac{1}{x^3}$ with common denominator.

21. Simplify $\frac{x^2 - y^2}{x^3 - y^3}$.
22. $\frac{1}{x^2 + 5x + 6} + \frac{1}{x + 2}$.
23. $\frac{4x}{x^2 - 9} + \frac{2}{x - 3}$.
24. Simplify $\frac{2x^3 - 2x}{4x^2 - 4}$.
25. Simplify $\frac{x + 1}{x^2 - x - 2} + \frac{x - 2}{x^2 - 4}$.

Advanced Practice (培优) · 10

Advanced Practice

1. Prove $\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$ and sum $\sum_{k=1}^n \frac{1}{k(k+1)}$.
2. Simplify $\frac{x^2 + (a+b)x + ab}{x^2 + (b+c)x + bc}$.
3. Write $\frac{x-a}{x-b} - \frac{x-b}{x-a}$ as a single fraction and simplify.
4. Write $\frac{5x+3}{(x+1)(x+2)}$ as $\frac{A}{x+1} + \frac{B}{x+2}$.
5. Simplify $\frac{\frac{1}{x} + \frac{1}{y}}{\frac{1}{x} - \frac{1}{y}}$.
6. Decompose $\frac{3x+5}{(x-1)(x+2)^2}$ into $\frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$.
7. Simplify $\frac{x^3+1}{x^2-x+1} \cdot \frac{x-1}{x^2-1}$.
8. Solve $\frac{x^2-1}{x-1} > 3$ for real x ($x \neq 1$).
9. Simplify $\frac{1}{x^2-3x+2} - \frac{1}{x^2-x} - \frac{1}{x^2-2x}$.

10. If $y = \frac{x+1}{x-1}$, express $\frac{y+1}{y-1}$ in terms of x only.

7 Equations & Inequalities (方程与不等式)

Linear and quadratic equations, simultaneous equations, and inequalities.

Rules 规则

- Collect the x terms on one side, then divide by the coefficient.
- For a quadratic equation, try factorising first. If it does not factorise, use the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ or complete the square.
- Solve simultaneous equations (联立) by elimination or substitution.
- Inequalities (不等式): when multiplying or dividing by a negative number, reverse the inequality sign.
- For $(x - a)(x - b) > 0$, use a sign chart or the critical points method.

Worked Examples 例题

Example 1. Solve $3x - 5 = 10$.

Add 5: $3x = 15$, so $x = 5$.

Example 2. Solve $x^2 - 5x + 6 = 0$.

Factorising: $(x - 2)(x - 3) = 0$, so $x = 2$ or $x = 3$.

Example 3. Solve $\begin{cases} x + y = 5 \\ 2x - y = 4 \end{cases}$.

Adding the equations: $3x = 9$, so $x = 3$ and $y = 2$.

Example 4. Solve $2x - 3 < 7$.

Add 3: $2x < 10$, so $x < 5$.

Basic Drill (入门综合) · 50

1. Solve $x + 4 = 11$.
2. Solve $2x = 18$.
3. Solve $3x - 1 = 8$.
4. Solve $5x + 2 = 17$.
5. Solve $7 - x = 3$.
6. Solve $\frac{x}{4} = 5$.
7. Solve $\frac{2x}{3} = 6$.
8. Solve $2(x + 3) = 14$.
9. Solve $3(x - 1) = 2x + 5$.
10. Solve $4x - 3 = 2x + 9$.
11. Solve $x^2 = 16$.
12. Solve $x^2 - 9 = 0$.
13. Solve $x^2 - 5x = 0$.
14. Solve $x^2 + 4x + 3 = 0$.
15. Solve $x^2 - 7x + 10 = 0$.
16. Solve $x^2 + x - 12 = 0$.
17. Solve $2x^2 - 5x - 3 = 0$.
18. Solve $x^2 - 2x - 1 = 0$ (surds).
19. Solve $x^2 + 6x + 9 = 0$.
20. Solve $(x - 4)(x + 1) = 0$.
21. $x + y = 10$, $x - y = 2$. Find x, y .
22. $2x + y = 7$, $x - y = 2$.
23. $3x + 2y = 12$, $x + y = 5$.
24. $y = 2x$, $x + y = 9$.
25. $y = x + 1$, $2x + 3y = 16$.
26. Solve $x > 3$. (describe)
27. Solve $x \leq -1$.
28. Solve $2x + 1 < 9$.
29. Solve $5 - 3x \geq 2$.
30. Solve $x + 4 > 2x$.
31. Solve $-2x > 6$.
32. Solve $x^2 - 4 \geq 0$ using critical points.
33. Solve $(x - 1)(x - 4) < 0$.
34. Solve $x^2 > 9$.

35. Solve $|x| = 5$.

43. Rearrange $y = mx + c$ for x .

36. Solve $|x - 2| = 3$.

44. Rearrange $s = \frac{1}{2}(u + v)t$ for u .

37. Solve $\frac{x+1}{2} = 4$.

45. Solve $x^2 - x - 6 = 0$.

38. Solve $0.5x = 7$ as fraction.

46. $x + 2y = 8, 2x + y = 7$.

39. Solve $3x^2 = 12$.

47. Solve $4x - 1 > 3x + 5$.

40. Solve $x(x - 5) = 0$.

48. Solve $1 - x \leq 4$.

41. Rearrange $v = u + at$ for t .

49. Solve $x^2 + 5x + 4 = 0$.

42. Rearrange $A = \pi r^2$ for r ($r > 0$).

50. Solve $2x^2 + x - 1 = 0$.

Practice (综合练习) · 25**Practice**

1. Solve $3(2x - 1) - 2(x + 4) = 5$.

2. Solve $\frac{x+3}{2} - \frac{x-1}{3} = 2$.

3. Solve $x^2 - 4x - 1 = 0$.

4. Solve $2x^2 + 3x - 4 = 0$.

5. Solve by completing square: $x^2 + 6x - 2 = 0$.

6. $2x + 3y = 13, 3x - y = 3$.

7. $x^2 + y = 5, x + y = 3$ (sub).

8. Solve $x^2 - 5x + 6 > 0$.

9. Solve $\frac{2x-1}{x+2} = 3$.

10. Solve $|2x - 1| = 5$.
11. Solve $|x + 3| < 2$.
12. Solve $\frac{1}{x} + \frac{1}{x+1} = \frac{5}{6}$.
13. Find k so $x^2 + kx + 9 = 0$ has equal roots.
14. Find k so $x^2 - 2x + k = 0$ has two distinct real roots.
15. Solve $y = x^2 - 1$, $y = 2x + 2$.
16. Solve $\frac{x}{x-2} + \frac{1}{x} = 3$ (state exclusions).
17. Solve $\frac{2}{x-2} + \frac{3}{x+2} = 1$ (state exclusions).
18. Solve $(2x - 1)(x + 3) \geq 0$.
19. Simultaneous: $xy = 12$, $x + y = 7$.
20. Find k so that one root of $x^2 + (k - 1)x + k = 0$ is double the other.
21. Solve $4^x = 8$.
22. Solve $\sqrt{2x + 3} = x$ and check.
23. Solve $|x^2 - 9| \leq 5$.
24. Solve $(x - 1)^2 < 2x + 7$.
25. Solve $|x - 2| + |x + 2| \leq 4$.

Advanced Practice (培优) · 10

Advanced Practice

1. Solve $x^4 - 5x^2 + 4 = 0$ by substituting $u = x^2$.
2. For which m does $x^2 - mx + m = 0$ have real roots? Give inequality for m .
3. Solve $x + y + z = 6$, $x + 2y + 3z = 14$, $x + 4y + 9z = 36$.
4. Solve $\frac{x^2 - 1}{x - 1} > 3$ ($x \neq 1$).
5. Prove that if $ax^2 + bx + c = 0$ has roots r, s , then $r + s = -b/a$, $rs = c/a$.
6. Solve $|x - 1| + |x + 2| = 5$ by cases.
7. Find all real solutions of $x^2 + (x - 1)^2 = (x + 1)^2$.
8. Solve $(k - 1)x = k + 2$ and discuss the case $k = 1$.
9. Find all real k such that $x^2 - 2x + k > 0$ for every real x .
10. Solve $x^2 + y^2 = 25$, $xy = 12$.

8 Quadratic Functions (一元二次函数)

Parabolas, completing the square, vertex, intercepts, and max/min values.

Rules 规则

- The graph of $y = ax^2 + bx + c$ is a parabola (抛物线). It opens upward if $a > 0$ and downward if $a < 0$.
- The axis of symmetry is given by $x = -\frac{b}{2a}$.
- Completing the square writes the function as $a(x - h)^2 + k$, and the vertex (顶点) is (h, k) .
- The y -intercept is c . To find the x -intercepts, solve $ax^2 + bx + c = 0$.
- In completed-square form, the maximum or minimum value is k .

Worked Examples 例题

Example 1. Find the intercepts and vertex of $y = x^2 - 4x + 3$.

x -intercepts: $(1, 0)$ and $(3, 0)$; y -intercept: $(0, 3)$; vertex: $(2, -1)$.

Example 2. Express $y = 2x^2 - 12x + 11$ in the form $2(x - h)^2 + k$.

The result is $2(x - 3)^2 - 7$; the minimum value is -7 .

Example 3. The range of $y = -(x - 1)^2 + 4$ is $y \leq 4$.

Basic Drill (入门综合) · 50

1. Does $y = x^2 + 1$ open up or down?
2. Does $y = -2x^2 + 3$ open up or down?
3. Find y -intercept of $y = x^2 - 5x + 7$.
4. Find x -intercepts of $y = x^2 - 5x + 6$.
5. Find axis of symmetry of $y = x^2 - 6x + 1$.
6. Find axis of $y = 2x^2 + 8x - 3$.
7. Vertex of $y = (x - 3)^2 + 2$.
8. Vertex of $y = -(x + 1)^2 - 5$.
9. Complete square: $x^2 + 6x + 5$.
10. Complete square: $x^2 - 4x + 1$.
11. Complete square: $x^2 + 2x - 7$.
12. Min value of $y = x^2 + 4x + 9$.
13. Max value of $y = -x^2 + 6x$.
14. Range of $y = (x - 2)^2 + 3$ for $x \in \mathbb{R}$.
15. Range of $y = -(x + 1)^2 + 8$.
16. Sketch labels: intercepts of $y = x^2 - x - 2$.
17. Vertex of $y = x^2 - 2x - 8$ by formula.
18. Write $y = x^2 + 10x + 21$ in completed-square form.
19. For $y = 2(x - 1)^2 + 5$, state vertex and min.
20. For $y = -3(x + 2)^2 + 7$, state vertex and max.
21. Find c if $y = x^2 + 4x + c$ passes through $(1, 8)$.
22. Does $y = x^2 + 1$ meet the x -axis?
23. Discriminant of $x^2 - 4x + 5$: how many x -intercepts?
24. Discriminant of $x^2 - 4x + 4$.
25. Discriminant of $x^2 - 4x + 3$.
26. Axis of $y = -x^2 + 5x$.
27. Vertex of $y = x^2$.
28. Vertex of $y = (x + 4)^2$.
29. Expand to standard form: $y = (x - 2)^2 + 3$.
30. Expand: $y = 2(x + 1)^2 - 5$.
31. Find y when $x = 0$ for $y = 3x^2 - x + 2$.
32. Find y when $x = 2$ for $y = x^2 - 3x$.

33. Complete square: $2x^2 + 8x + 3$ (factor 2 first).
34. Complete square: $-x^2 + 2x + 5$.
35. State turning point type (max/min) for $y = 5x^2 - x$.
36. State turning point type for $y = 1 - x^2$.
37. Find intercepts of $y = (x + 3)(x - 1)$.
38. Find intercepts of $y = -(x - 2)(x - 6)$.
39. Line of symmetry for $y = (x - 5)(x + 1)$.
40. Min of $y = x^2 - 8x + 20$.
41. Max of $y = 4x - x^2$.
42. Range of $y = x^2 + 1$.
43. Range of $y = 3 - (x + 2)^2$.
44. If vertex is $(2, -3)$ and $a = 1$, write a possible equation.
45. If vertex $(0, 4)$ and passes through $(1, 5)$, find a in $y = ax^2 + 4$.
46. Compare widths: $y = x^2$ vs $y = 3x^2$ (qualitative).
47. For $y = x^2 - 6x + k$, find k so vertex on x -axis.
48. Find coordinates of minimum point of $y = x^2 + x + 1$.
49. Find coordinates of maximum point of $y = -2x^2 + 8x - 1$.
50. Write $y = x^2 - x - \frac{3}{4}$ in completed-square form.

Practice (综合练习) · 25

Practice

- Write $f(x) = 3x^2 - 12x + 7$ as $a(x - h)^2 + k$; state vertex and range.
- Sketch $y = x^2 - 4x - 5$ with intercepts and vertex.
- Find max value of $y = 7 + 6x - x^2$ and where it occurs.
- Find the range of $g(x) = -x^2 + 4x - 1$ on $x \in [0, 5]$.
- For $y = x^2 + bx + c$ with vertex $(3, -4)$, find b, c .

6. Find k so $y = x^2 - 2x + k$ touches the x -axis.
7. Intersection: solve $y = x^2 - 3x$ and $y = x + 2$; interpret geometrically.
8. $f(x) = (x - a)^2 + b$ has minimum 5 at $x = -2$. Find a, b .
9. Show $x^2 + 4x + 7 > 0$ for all real x .
10. Find the axis and vertex of $y = 3x^2 - 5x + 1$.
11. Complete square for $y = -2x^2 + 8x - 3$; state max.
12. A parabola passes through $(0, 3)$, $(1, 0)$, $(2, 1)$. Find equation (system).
13. For $y = ax^2 + bx + c$, given axis $x = 2$, y -intercept 5, and passes through $(1, 2)$, find a, b, c .
14. Find range of $y = x^2 - 2x$ for $x \in [0, 3]$.
15. Translate $y = x^2$ to vertex $(h, k) = (4, -1)$; write equation.
16. Write $y = -3x^2 + 12x - 5$ in vertex form and find its maximum on $[0, 4]$.
17. For which m does $y = x^2 + mx + 1$ have two distinct x -intercepts?
18. Write $y = 4x^2 - 4x + 7$ in the form $a(x - h)^2 + k$.
19. Find the intersection points of $y = x^2 + 1$ and $y = 2x + 2$.
20. Prove that the minimum of $ax^2 + bx + c$ ($a > 0$) is $\frac{4ac - b^2}{4a}$.
21. A ball path $h = -t^2 + 6t + 1$. Find the maximum height.
22. Parabola through $(-1, 0)$ and $(3, 0)$ with vertex $(1, 8)$: find its equation.
23. Find p so the vertex of $y = x^2 + px + p$ lies on $y = x$.
24. State vertex, axis, and max value of $y = -(x + 3)^2 + 5$.

25. Factorise $y = 2x^2 - x - 6$; find intercepts and axis of symmetry; hence find the range on $x \in [-1, 2]$.

Advanced Practice (培优) · 10

Advanced Practice

1. Find the shortest distance from $(0, 3)$ to the curve $y = x^2$.
2. For $f(x) = x^2 + 2mx + m^2 + 1$, show $f(x) \geq 1$ for all real x and all real m .
3. Find all k such that $y = x^2 - kx + k$ has its vertex in the first quadrant.
4. The curves $y = x^2 + ax + b$ and $y = 1 - x^2$ touch. Find the relation between a and b .
5. Find the range of $f(x) = x + \frac{1}{x}$ for $x > 0$.
6. Complete the square: $x^2 + y^2 + 4x - 6y$.
7. Parabola $y = x^2 + bx + c$ has x -intercepts 1 unit from the axis of symmetry. If $c = 3$, find b .
8. Show that $x^2 + (1 - x)^2 \geq \frac{1}{2}$ for all real x .
9. For $y = x^2 - 2cx + c$, find the locus of the vertex as c varies.
10. Find the smallest real k such that $x^2 - 4x + 1 \leq k$ for all $x \in [-1, 4]$.

9 Coordinate Geometry (坐标几何)

Gradient, midpoint, distance, straight-line equations, and parallel and perpendicular lines.

Rules 规则

- Gradient (斜率): $m = \frac{y_2 - y_1}{x_2 - x_1}$.
- Midpoint: $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
- Distance: $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- Equation of a straight line: $y - y_1 = m(x - x_1)$, $y = mx + c$, $ax + by + c = 0$.
- Parallel lines have equal gradients; perpendicular lines satisfy $m_1 m_2 = -1$.

Worked Examples 例题

Example 1. The gradient of AB with $A(1, 2)$ and $B(5, 10)$ is $\frac{10 - 2}{5 - 1} = \frac{8}{4} = 2$.

Example 2. The midpoint of $(2, -3)$ and $(6, 5)$ is $(4, 1)$, and the distance between them is $\sqrt{4^2 + 8^2} = \sqrt{80} = 4\sqrt{5}$.

Example 3. The line through $(1, 4)$ with gradient -2 is $y - 4 = -2(x - 1)$, giving $y = -2x + 6$.

Example 4. A line perpendicular to $y = 3x + 1$ has gradient $-\frac{1}{3}$.

Basic Drill (入门综合) · 50

1. Gradient $A(0, 0)$, $B(4, 2)$.
2. Gradient $A(1, 3)$, $B(5, 11)$.
3. Gradient $A(2, 7)$, $B(6, 7)$.
4. Gradient $A(3, 1)$, $B(3, 8)$ (state).
5. Gradient $A(-1, 4)$, $B(3, -2)$.
6. Midpoint $(0, 0)$ & $(6, 4)$.
7. Midpoint $(2, 5)$ & $(8, 1)$.
8. Midpoint $(-3, 4)$ & $(5, -2)$.
9. Distance $(0, 0)$ to $(3, 4)$.
10. Distance $(1, 2)$ to $(4, 6)$.
11. Distance $(-1, 0)$ to $(3, 3)$.
12. Equation: through $(0, 2)$ gradient 3.
13. Equation: through $(0, -1)$ gradient -2 .
14. Equation: through $(1, 1)$ gradient 1.
15. Equation: through $(2, 3)$ gradient $-\frac{1}{2}$.
16. Find c if $y = 2x + c$ passes through $(3, 10)$.
17. $y = 4x - 1$: gradient and y -intercept.
18. $2x + y - 5 = 0$: write $y = mx + c$.
19. $3x - 2y + 6 = 0$: find m .
20. Parallel to $y = 5x + 2$: give any line.
21. Perpendicular to $y = 2x$: give gradient.
22. Perpendicular to $y = -\frac{1}{3}x + 4$: gradient.
23. Do $y = 2x + 1$ and $y = 2x - 5$ parallel?
24. Do $y = x$ and $y = -x + 3$ perpendicular?
25. Line through $(0, 0)$ and $(2, 6)$: equation.
26. Line through $(1, 2)$ and $(3, 8)$.
27. Line through $(0, 4)$ and $(4, 0)$.
28. Find gradient of segment joining $(a, 0)$ and $(0, a)$ ($a \neq 0$).
29. Midpoint of (a, b) and (c, d) .
30. Distance formula for (a, b) to origin.
31. x -intercept of $y = 2x - 8$.
32. y -intercept of $3x + y = 9$.
33. Point-slope: $(4, -1)$, $m = 2$.

34. Convert $y = \frac{1}{2}x + 3$ to $ax + by + c = 0$ integers.
35. Is $(2, 5)$ on $y = 3x - 1$?
36. Is $(1, 1)$ on $2x + 3y - 5 = 0$?
37. Vertical line through $(3, 7)$: equation.
38. Horizontal line through $(3, 7)$.
39. Gradient between $(1, -2)$ and $(-3, 6)$.
40. Find k so gradient from $(1, 2)$ to $(k, 8)$ equals 2.
41. Find k so midpoint of $(k, 1)$ and $(3, 5)$ is $(2, 3)$.
42. Distance from $(1, 1)$ to $(1, 1)$.
43. Points $(0, 0)$, $(4, 0)$, $(0, 3)$ form a right triangle. Find the hypotenuse length.
44. Parallel to x -axis through $(2, -5)$.
45. Parallel to y -axis through $(2, -5)$.
46. Gradient of $y = 7$.
47. Equation with gradient 0 through $(5, 3)$.
48. Find m if $y = mx + 1$ perpendicular to $y = \frac{1}{4}x$.
49. Line through $(2, 2)$ parallel to $y = 3x - 1$.
50. Line through $(2, 2)$ perpendicular to $y = 3x - 1$.

Practice (综合练习) · 25**Practice**

- Find the equation of the line through $(1, -3)$ perpendicular to $2x + y = 5$, in the form $ax + by + c = 0$.
- $A(2, 1)$, $B(6, 7)$, $C(0, 9)$. Decide whether $AB \perp AC$ by comparing gradients.
- Midpoint M of $A(3, -2)$ and $B(7, 4)$. Find the line through M parallel to $y = x$.
- Triangle $A(0, 0)$, $B(6, 0)$, $C(2, 4)$. Find the equation of the median from C .
- Show that $(0, 0)$, $(6, 0)$, $(3, 3\sqrt{3})$ form an equilateral triangle.
- Find p so that $(p, 3)$ is equidistant from $(0, 0)$ and $(6, 0)$.

7. Find the intersection of $y = 2x - 1$ and $x + 2y = 8$, then the distance from this point to the origin.
8. Solve the system $x + 2y = 8$, $3x - y = 3$. Find the area of the triangle with vertices $(0, 0)$, the x -intercept of $x + 2y = 8$, and the intersection point.
9. Area of triangle with vertices $(0, 0)$, $(4, 0)$, $(1, 3)$.
10. Find the perpendicular bisector of $(2, 5)$ and $(6, 1)$.
11. Decide whether $A(1, 2)$, $B(3, 6)$, $C(7, 5)$ is right-angled at B .
12. Lines $2x + y = 5$ and $x - 2y = 3$ meet at P . Find the equation of the line through P with gradient $-\frac{1}{2}$.
13. Find an equation $ax + by + c = 0$ for the line through $(1, 1)$ and $(2, -1)$.
14. Find k so that the line through $(1, 4)$ and $(k, 0)$ is perpendicular to $y = 2x$.
15. A line has inclination 60° and passes through $(0, 1)$. Write its equation.
16. Point dividing $(1, 2)$ to $(7, 8)$ in the ratio $1 : 2$.
17. Find t so that $(1, 1)$, $(2, 3)$, $(3, t)$ are collinear.
18. Find k so that the distance from $(0, 0)$ to (k, k) is $5\sqrt{2}$.
19. Line with intercepts 3 and 6: write $\frac{x}{a} + \frac{y}{b} = 1$ and $ax + by + c = 0$.
20. Foot of the perpendicular from $(0, 0)$ to $x + y - 2 = 0$.
21. $A(0, 0)$, $B(3, 4)$, $C(5, 1)$. Find D so $ABCD$ is a parallelogram with $\overrightarrow{AB} = \overrightarrow{DC}$.
22. The line through $(2, 3)$ meets the axes at P and Q . If PQ has midpoint $(2, 3)$, find the intercepts.
23. Find the acute angle between $y = x$ and $y = \sqrt{3}x$ using $\tan$.

24. Locus of points equidistant from $(0, 0)$ and $(4, 0)$: state the equation.
25. Find the area of the quadrilateral with vertices $(0, 0)$, $(4, 0)$, $(5, 2)$, $(1, 3)$.

Advanced Practice (培优) · 10**Advanced Practice**

1. Using midpoints, prove that the diagonals of a parallelogram bisect each other.
2. On the x -axis, find P minimising $PA + PB$ for $A(1, 4)$ and $B(5, 2)$.
3. On $y = x^2$, find the equation of the chord joining the points with $x = 1$ and $x = 2$.
4. Find c so that $y = x + 1$, $y = 2x - 1$ and $y = -x + c$ are concurrent.
5. Prove A, B, C are collinear iff $\overrightarrow{AB} = k\overrightarrow{AC}$ for some scalar k .
6. Find both lines of gradient 1 at distance 2 from the origin.
7. Midpoints of $A(0, 0)$, $B(2a, 0)$, $C(2b, 2c)$, $D(0, 2d)$ form a parallelogram. Prove it.
8. Among lines $\frac{x}{a} + \frac{y}{b} = 1$ through $(1, 2)$, minimise $a + b$ ($a, b \neq 0$).
9. Distance between $x + y = 1$ and $x + y = 5$.
10. $A(1, 0)$, $B(0, 1)$, P on the segment of $x + y = 2$ in the first quadrant. Maximise $PA^2 + PB^2$.

10 Functions & Graphs (函数与图像)

Function notation, domain and range, shifts and stretches.

Rules 规则

- Function notation: $y = f(x)$, where x is the input and $f(x)$ is the output.
- Domain (定义域): the set of allowed x values. Range (值域): the set of possible outputs.
- $y = f(x) + k$ shifts the graph k units up, and $y = f(x - a)$ shifts it a units to the right.
- $y = cf(x)$ stretches the graph vertically by a factor of $|c|$, and $y = f(cx)$ stretches it horizontally by a factor of $\frac{1}{|c|}$.
- $y = -f(x)$ reflects the graph in the x -axis, and $y = f(-x)$ reflects it in the y -axis.

Worked Examples 例题

Example 1. If $f(x) = 2x - 3$, then $f(5) = 7$ and $f(-1) = -5$.

Example 2. The domain of $f(x) = \sqrt{x - 2}$ is $x \geq 2$, and its range is $f(x) \geq 0$.

Example 3. Starting from $y = x^2$, the graph of $y = (x - 3)^2 + 1$ is translated 3 units right and 1 unit up.

Example 4. The graph of $y = 2f(x)$ is a vertical stretch of $y = f(x)$ by a factor of 2.

Basic Drill (入门综合) · 50

1. If $f(x) = 3x + 1$, find $f(2)$.
2. If $f(x) = x^2 - 4$, find $f(3)$.
3. If $f(x) = \frac{1}{x}$, find $f(5)$.
4. If $f(x) = 2x - 5$, find x with $f(x) = 0$.
5. Domain of $f(x) = \frac{1}{x-4}$.
6. Domain of $f(x) = \sqrt{x+3}$.
7. Domain of $f(x) = \frac{1}{\sqrt{x}}$.
8. Range of $f(x) = x + 2$ for $x \in \mathbb{R}$.
9. Range of $f(x) = x^2$ for $x \in \mathbb{R}$.
10. Range of $f(x) = |x|$.
11. Evaluate $f(0)$ for $f(x) = 5 - x^2$.
12. If $f(x) = x^2$, write $f(x) + 3$.
13. If $f(x) = x^2$, write $f(x-2)$.
14. Describe shift: $y = x^2 + 4$.
15. Describe shift: $y = (x+5)^2$.
16. Describe shift: $y = (x-1)^2 - 2$.
17. Describe: $y = -x^2$ from $y = x^2$.
18. Describe: $y = f(-x)$ vs $y = f(x)$.
19. If $f(x) = x$, describe the graph of $y = 2f(x)$.
20. If $f(x) = x$, describe the graph of $y = f(2x)$.
21. $f(x) = 3x$, find $f(a+1)$.
22. $f(x) = x^2 + x$, find $f(-2)$.
23. Domain of $\frac{x}{x^2-1}$.
24. Which inputs forbidden for $\sqrt{5-x}$?
25. If $f(x) = 5$, constant: range?
26. Does $y^2 = x$ define y as a single function of x ? (Yes/No).
27. Table: $x = 1, 2, 3$ and $f(x) = 2x$; list outputs.
28. If $f(2) = 7$ and linear $f(x) = mx + c$ with $f(0) = 1$, find f .
29. Describe $y = |x-2|$ relative to $y = |x|$.
30. Describe $y = |x| - 2$.
31. From $y = \sqrt{x}$ to $y = \sqrt{x-4}$.

32. From $y = \sqrt{x}$ to $y = \sqrt{x} + 3$.
33. Stretch: $y = 3x^2$ vs $y = x^2$.
34. Compress in x : $y = (2x)^2$ vs $y = x^2$.
35. $f(x) = 1 - x$. Find $f(f(2))$.
36. Is every straight line a function $y = f(x)$?
Exception?
37. Domain/range of $f(x) = 4 - x$ on $x \in [0, 4]$.
38. Find $f(3)$ if $f(x) = x^3$.
39. $f(x) = x$ for $x \geq 0$ and $f(x) = -x$ for $x < 0$. Name this function.
40. Explain briefly why each input x must give exactly one output for $y = f(x)$.
41. If $f(x) = 2x + 1$, solve $f(x) = f(5)$.
42. If $f(x) = x^2$, solve $f(x) = 9$.
43. Vertical line test: can a circle be $y = f(x)$ for all x on it?
44. Write formula for shift of $y = 1/x$ right by 1.
45. Write formula for shift of $y = 1/x$ up by 2.
46. $f(x) = x^2 - 1$: find coordinates where graph meets axes.
47. Describe $y = -f(x) + 1$ if you know $y = f(x)$.
48. Domain of $f(x) = \frac{1}{x^2 + 1}$ (all reals?).
49. Range of $f(x) = \frac{1}{x^2 + 1}$.
50. If $g(x) = f(x - 3) + 2$ and $f(0) = 5$, find $g(3)$.

Practice (综合练习) · 25

Practice

1. State domain and range of $f(x) = \sqrt{9 - x^2}$.
2. State domain and range of $g(x) = \frac{2x + 1}{x - 3}$.
3. Describe fully the transformation from $y = f(x)$ to $y = 2f(x - 1) + 3$.
4. From $y = x^2$ obtain equation after: reflect in x -axis, then shift up 4.
5. If $f(x) = x^2 - 2x$, find and simplify $f(x + 1) - f(x)$.

6. Find domain of $f(x) = \frac{\sqrt{x+2}}{x-1}$.
7. Range of $f(x) = (x-1)^2 + 2$ for $x \in \mathbb{R}$.
8. Range of $f(x) = 1 - \sqrt{x}$ for $x \geq 0$.
9. Explain why $y = \pm\sqrt{x}$ is not a function, but $y = \sqrt{x}$ is.
10. Given graph of $y = f(x)$ through $(1, 2)$, where does $y = f(x) + 3$ send that point?
11. Point $(1, 2)$ on $y = f(x)$; image under $y = f(2x)$.
12. Point $(1, 2)$ on $y = f(x)$; image under $y = f(x/2)$.
13. If $f(x) = |x-1| + |x+1|$, evaluate $f(0), f(2), f(-3)$ and sketch piecewise.
14. Find the domain of $f(x) = \frac{1}{\sqrt{3-2x}}$.
15. Transform $y = \frac{1}{x}$ to $y = \frac{1}{x-2} + 1$; asymptotes?
16. If $f(x) = x^2$ for $x \geq 0$, find $f^{-1}(x)$ and state the domain and range of f^{-1} .
17. If $f(x) = 2x - 3$, find expression for the inverse idea: solve $y = 2x - 3$ for x .
18. Sequence of transforms: $y = x^2 \rightarrow y = (x+2)^2 \rightarrow y = -(x+2)^2 \rightarrow y = -(x+2)^2 + 5$.
19. Find a so domain of $\sqrt{x^2 - a}$ is $|x| \geq 2$.
20. Given $f(x) = x^2 - 1$, solve $f(f(x)) = 0$.
21. Range of $f(x) = x + \frac{1}{x}$ for $x \geq 1$.
22. Describe the key features of $y = |x^2 - 1|$ (intercepts and turning behaviour).
23. Decide whether $x = y^2$ defines y as a function of x .
24. Given $f(x) = x^2$ and $g(x) = x + 1$, compute $f(g(2))$ and $g(f(2))$.

25. Explain briefly why $y = -\sqrt{-x}$ is not just a stretch or reflection of $y = \sqrt{x}$.

Advanced Practice (培优) · 10

Advanced Practice

- Find range of $f(x) = x^2 - 4x$ on $[-1, 5]$ by vertex vs endpoints.
- Solve $f(x) = f^{-1}(x)$ for $f(x) = 2x + 1$ after finding inverse expression.
- Show $f(x) = \frac{x}{x-1}$ satisfies $f(f(f(x))) = x$ for allowed x .
- Write $y = 2 - 3f(1 - x)$ as a sequence of elementary transforms on $y = f(x)$.
- Domain of $\sqrt{x} + \sqrt{1-x}$ and range (exact).
- For $f(x) = ax + b$, find condition that $f \circ f = f$ (projection functions).
- Piecewise $f(x) = \begin{cases} x+2 & x < 0 \\ x^2 & x \geq 0 \end{cases}$: is it continuous at 0? range on $[-2, 2]$?
- Find all linear $f(x) = mx + c$ with $f(f(x)) = x$ for all x (involutions).
- The curve $y = \frac{x^2 + 1}{x}$ has an oblique asymptote. State its equation and justify briefly.
- Find the range of $\frac{x^2 + 1}{x^2 + x + 1}$ for real x .

11 Trigonometry (三角)

Exact values, radians, and simple trig equations.

Rules 规则

- Exact values (精确值): $\sin 30^\circ = \frac{1}{2}$, $\cos 30^\circ = \frac{\sqrt{3}}{2}$, $\tan 30^\circ = \frac{1}{\sqrt{3}}$; $\sin 45^\circ = \cos 45^\circ = \frac{\sqrt{2}}{2}$, $\tan 45^\circ = 1$; $\sin 60^\circ = \frac{\sqrt{3}}{2}$, $\cos 60^\circ = \frac{1}{2}$, $\tan 60^\circ = \sqrt{3}$.
- Radians (弧度): $\pi \text{ rad} = 180^\circ$.
- $\sin^2 \theta + \cos^2 \theta = 1$, $\tan \theta = \frac{\sin \theta}{\cos \theta}$.
- Use the CAST diagram or the unit circle to determine the sign of a ratio; solve simple trig equations by following standard patterns.

Worked Examples 例题

Example 1. $\sin 60^\circ = \frac{\sqrt{3}}{2}$, $\tan 45^\circ = 1$.

Example 2. Convert 60° to radians: $\frac{\pi}{3}$.

Example 3. Solve $\sin \theta = \frac{1}{2}$ for $0 \leq \theta \leq 180^\circ$: $\theta = 30^\circ$ or 150° .

Example 4. Verify the identity: $\cos^2 30^\circ + \sin^2 30^\circ = 1$.

Basic Drill (入门综合) · 50

1. $\sin 30^\circ$.
2. $\cos 30^\circ$.
3. $\tan 30^\circ$.
4. $\sin 45^\circ$.
5. $\cos 45^\circ$.
6. $\tan 45^\circ$.
7. $\sin 60^\circ$.
8. $\cos 60^\circ$.
9. $\tan 60^\circ$.
10. $\sin 0^\circ, \cos 0^\circ$.
11. $\sin 90^\circ, \cos 90^\circ$.
12. Convert 90° to radians.
13. Convert 180° to radians.
14. Convert 45° to radians.
15. Convert 30° to radians.
16. Convert $\frac{\pi}{2}$ to degrees.
17. Convert $\frac{\pi}{6}$ to degrees.
18. Convert $\frac{2\pi}{3}$ to degrees.
19. Evaluate $\sin \frac{\pi}{6}$.
20. Evaluate $\cos \frac{\pi}{3}$.
21. Evaluate $\tan \frac{\pi}{4}$.
22. $\sin^2 30^\circ + \cos^2 30^\circ$.
23. $\tan 60^\circ \cdot \tan 30^\circ$.
24. Find $\sin \theta$ if $\cos \theta = \frac{3}{5}$ acute.
25. Find $\tan \theta$ if $\sin \theta = \frac{5}{13}$ acute.
26. Sign of $\sin 120^\circ$.
27. Sign of $\cos 120^\circ$.
28. Sign of $\tan 120^\circ$.
29. $\sin 150^\circ$ exact.
30. $\cos 150^\circ$ exact.
31. $\sin 180^\circ$.
32. $\cos 180^\circ$.
33. Solve $\sin \theta = 0$ for $0 \leq \theta \leq 180^\circ$.
34. Solve $\cos \theta = 0$ for $0 \leq \theta \leq 180^\circ$.

35. Solve $\tan \theta = 1$ for $0 \leq \theta \leq 180^\circ$.
36. Solve $\sin \theta = \frac{\sqrt{3}}{2}$ for $0 \leq \theta \leq 180^\circ$.
37. Solve $\cos \theta = \frac{1}{2}$ for $0 \leq \theta \leq 180^\circ$.
38. Is $\sin \theta = \cos \theta$ at 45° ? Check.
39. $2 \sin \theta = 1$ with $0 \leq \theta \leq 180^\circ$.
40. Evaluate $2 \cos 30^\circ$.
41. Evaluate $\sqrt{2} \sin 45^\circ$.
42. Right triangle: opposite 3, hypotenuse 5; $\sin$.
43. Adjacent 4, hypotenuse 5; $\cos$.
44. Opposite 3, adjacent 4; $\tan$.
45. $\sin(90^\circ - \theta) = \cos \theta$ —use $\theta = 30^\circ$ check.
46. Convert 270° to radians.
47. Evaluate $\cos \frac{\pi}{4}$ exact.
48. Evaluate $\sin \frac{\pi}{3}$ exact.
49. Which larger: $\sin 30^\circ$ or $\sin 60^\circ$?
50. Fill: $\tan \theta = \frac{\sin \theta}{?}$.

Practice (综合练习) · 25

Practice

- Solve $2 \sin \theta = \sqrt{3}$ for $0 \leq \theta < 360^\circ$.
- Solve $\cos \theta = -\frac{1}{2}$ for $0 \leq \theta < 360^\circ$.
- Solve $\tan \theta = -\sqrt{3}$ for $0 \leq \theta < 360^\circ$.
- Solve $2 \cos \theta = \sqrt{2}$ for $0 \leq \theta \leq 180^\circ$.
- Exact values of $\sin 225^\circ$ and $\cos 225^\circ$.
- Exact value of $\cos 315^\circ + \sin 315^\circ$.
- If $\sin \theta = \frac{3}{5}$ and θ obtuse, find $\cos \theta$ and $\tan \theta$.
- Solve $2 \sin \theta \cos \theta = \frac{\sqrt{3}}{2}$ for $0 \leq \theta \leq 180^\circ$.

9. Solve $\sin 2\theta = \frac{1}{2}$ for $0 \leq \theta \leq 180^\circ$.
10. In triangle ABC , $a = b = 4$ and $C = 60^\circ$. Find the area and side c .
11. Solve $\cos \theta = \cos 40^\circ$ for $0 \leq \theta < 360^\circ$.
12. Evaluate $\sin^2 45^\circ + \cos^2 30^\circ$ exactly.
13. In triangle ABC , side $b = 6$, angle $C = 30^\circ$, and the area is 9. Find side a .
14. Solve $\sin \theta = \frac{\sqrt{3}}{2}$ for $0 \leq \theta \leq \pi$.
15. Solve $\tan \theta = 1$ for $0 \leq \theta \leq 2\pi$, excluding where $\tan$ is undefined.
16. If $\tan \theta = 2$ and θ acute, find $\sin \theta$ in surd form.
17. Solve $4 \sin \theta = 2\sqrt{3}$ for $0 \leq \theta \leq 180^\circ$.
18. Find the acute θ with $\cos \theta = \sin 20^\circ$.
19. Exact value of $\tan 135^\circ$.
20. From a point 12 m from the base of a tower, the angle of elevation to the top is 30° . Find the height of the tower in exact form.
21. Solve $\sin \theta = \cos 2\theta$ for $0 \leq \theta \leq 180^\circ$ (use a double-angle identity).
22. Find all $\theta \in [0, 360^\circ)$ with $\sin \theta = \sin 20^\circ$.
23. Exact value of $\sin 15^\circ$ using $\sin(45^\circ - 30^\circ)$.
24. Show that $\sin(90^\circ - \theta) = \cos \theta$ and use it to evaluate $\sin 70^\circ$ in terms of $\cos$.
25. A sector of radius 6 has angle $\frac{\pi}{3}$. Find its arc length and area.

Advanced Practice (培优) · 10

Advanced Practice

1. Solve $\sin \theta = \cos \theta$ for $0 \leq \theta < 360^\circ$.
2. Solve $2 \sin^2 \theta - \sin \theta = 0$ for $0 \leq \theta \leq 360^\circ$.
3. Solve $\cos 2\theta = \frac{1}{2}$ for $0 \leq \theta \leq 180^\circ$.
4. Prove $\sin(90^\circ + \theta) = \cos \theta$ from the unit circle.
5. Find exact $\sin 15^\circ$ and $\cos 15^\circ$.
6. Write $3 \sin \theta + 4 \cos \theta$ as $R \sin(\theta + \alpha)$ and state the maximum value.
7. Solve $\tan \theta = \sin \theta$ for $0 \leq \theta < 360^\circ$.
8. If $\sin \theta + \cos \theta = 1$, find $\sin \theta \cos \theta$ and $\sin 2\theta$.
9. Solve $\sin \theta + \sqrt{3} \cos \theta = 1$ for $0 \leq \theta \leq 360^\circ$.
10. Show that $\sin \theta = \frac{p^2 - q^2}{p^2 + q^2}$, $\cos \theta = \frac{2pq}{p^2 + q^2}$ satisfy $\sin^2 \theta + \cos^2 \theta = 1$.

12 Exponentials & Logarithms (指数与对数)

Definitions, laws, and solving basic exponential equations.

Rules 规则

- $y = a^x$ ($a > 0, a \neq 1$); $y = \log_a x$ iff $a^y = x$ ($x > 0$).
- $\log_a(xy) = \log_a x + \log_a y$, $\log_a \frac{x}{y} = \log_a x - \log_a y$, $\log_a(x^k) = k \log_a x$.
- $\log_a a = 1$, $\log_a 1 = 0$, $a^{\log_a x} = x$.
- Change of base: $\log_a x = \frac{\log_b x}{\log_b a}$.
- To solve $a^x = b$, take logarithms or match the bases.

Worked Examples 例题

Example 1. $\log_2 8 = 3$ because $2^3 = 8$.

Example 2. $\log_3 9 + \log_3 3 = \log_3 27 = 3$.

Example 3. Solve $2^x = 32$: $x = 5$.

Example 4. Solve $3^x = 10$: $x = \frac{\ln 10}{\ln 3}$ (or $\log_3 10$).

Basic Drill (入门综合) · 50

1. Evaluate $\log_{10} 100$.
2. Evaluate $\log_{10} 1000$.
3. Evaluate $\log_2 16$.
4. Evaluate $\log_2 1$.
5. Evaluate $\log_5 5$.
6. Evaluate $\log_3 27$.
7. Evaluate $\log_4 2$.
8. Evaluate $\log_9 3$.
9. Evaluate $\log_8 1$.
10. Write $\log_2 8 = x$ as a power statement.
11. Solve $10^x = 1000$.
12. Solve $2^x = 8$.
13. Solve $5^x = 1$.
14. Solve $3^x = 81$.
15. Simplify $\log_2 4 + \log_2 8$.
16. Simplify $\log_3 9 - \log_3 3$.
17. Simplify $2\log_5 5$.
18. Simplify $\log_{10}(10^7)$.
19. Simplify $10^{\log_{10} 3}$.
20. Simplify $e^{\ln 5}$ (if $\ln$ used).
21. Domain: x in $\log_{10} x$.
22. Domain: $\log_2(x - 1)$.
23. Evaluate $\log_2 \frac{1}{8}$.
24. Evaluate $\log_3 \frac{1}{9}$.
25. Solve $2^x = \frac{1}{4}$.
26. Solve $10^x = 0.01$.
27. Rewrite $\log_a b = c$ as $a^c = b$.
28. Simplify $\log_a(a^4)$.
29. Simplify $\log_a \sqrt{a}$.
30. $\log_2 16 + \log_2 2$.
31. $\log_5 25 + \log_5 5$.
32. Expand $\log_{10}(xy)$ as sum.
33. Expand $\log_{10} \frac{x}{y}$.
34. Compress $\log_2 x + \log_2 y$ to one log.

35. Compress $3\log_2 x$ to one log.
36. Solve $\log_2 x = 5$.
37. Solve $\log_{10} x = 2$.
38. Solve $\log_3 x = -2$.
39. Evaluate $\log_4 8$ by change of base to 2.
40. Evaluate $\log_9 27$ via base 3.
41. Is $\log_2(-4)$ defined? (Yes/No).
42. $a^0 = 1$ link: $\log_a 1$.
43. Solve $4^x = 2^6$ by same base.
44. Solve $9^x = 3^4$.
45. Simplify $\ln e^3$.
46. Simplify $\log_b b^k$.
47. Compute $\log_2 2 + \log_2 2 + \log_2 2$.
48. If $\log_{10} 2 \approx 0.3010$, approx $\log_{10} 8$ (3sf).
49. Solve $5^x = 5^{2x-1}$ (exponents).
50. Write $32 = 2^x$ find x .

Practice (综合练习) · 25

Practice

1. Solve $2^{x+1} = 3^x$ leaving answer with logs.
2. Solve $\log_2(x-1) + \log_2(x+1) = 3$.
3. Solve $\log_3(2x-1) = 2$.
4. Solve $\log_3(x-1) + \log_3(x+1) = \log_3 8$ (state domain restrictions).
5. Prove $\log_a b \cdot \log_b a = 1$ ($a, b > 0, a, b \neq 1$).
6. Solve $4^x - 2^{x+1} - 8 = 0$ by setting $u = 2^x$.
7. Solve $\log_x 9 = 2$ ($x > 0, x \neq 1$).
8. Solve $\log_2(x^2 - 1) - \log_2(x - 1) = 3$ (state exclusions).
9. Solve $e^{2x} = 7$.

10. Solve $\ln(x + 2) = 0$.
11. If $\log_2 3 = a$ and $\log_2 5 = b$, express $\log_2 45$ in terms of a and b .
12. Solve $10^{2x} - 10^x - 6 = 0$ by setting $u = 10^x$.
13. Find the domain of $\log_2(x^2 - 5x + 6)$.
14. Solve $\log_5 x + \log_5(x - 4) = 1$.
15. Express $\log_{10} 12$ in terms of $\log_{10} 2$ and $\log_{10} 3$.
16. If $\log_2 3 = a$, write $\log_2 18$ in terms of a .
17. Solve $3^{2x} - 12 \cdot 3^x + 27 = 0$.
18. Solve $2\log_{10} x = \log_{10}(2x - 1)$.
19. Using $\log_{10} 2 \approx 0.3010$ and $\log_{10} 3 \approx 0.4771$, estimate $\log_{10} 6$.
20. Solve $(0.5)^x = 8$.
21. If $f(x) = 3^x$, write f^{-1} in logarithmic form.
22. Solve $\log_2 x = \log_4(x + 6)$.
23. Solve $2^{x+1} - 2^x = 24$.
24. Solve $5^x = 12$ to 3 significant figures.
25. Solve $2^x = 3^{x-1}$ leaving the answer in log form.

Advanced Practice (培优) · 10

Advanced Practice

1. Solve $\log_2 x + \log_x 2 = \frac{5}{2}$ ($x > 0, x \neq 1$).
2. Prove $\log_{a^n} b^m = \frac{m}{n} \log_a b$.
3. Find all $x > 0$ such that $x^x = x^2$.
4. Solve $\log_2(9 - 2^x) = 3 - x$.
5. Prove that $2^x + 2^{-x} \geq 2$ for all real x , with equality at $x = 0$.
6. Find the minimum value of $f(x) = 2^x + 2^{-x}$ for real x .
7. Solve system: $\log_2 x + \log_2 y = 3, x + y = 6$.
8. Verify that $\log_a b \cdot \log_b c \cdot \log_c a = 1$.
9. Solve $x^{\log_{10} x} = 100x$.
10. Solve $\log_2 x + \log_4 x + \log_{16} x = \frac{21}{4}$.