CAMBRIDGE

Sixth Term Examination Paper [STEP]

Mathematics 3 [9475]

2024

Examiners' Report

Mark Scheme

STEP MATHEMATICS 3

2024

Mark Scheme

1. (i)

$$\frac{1}{r+1}-\frac{1}{r}+\frac{1}{r^2}=\frac{1}{r^2(r+1)}[r^2-r(r+1)+(r+1)]$$

$$=\frac{1}{r^2(r+1)}[r^2-r^2-r+r+1]=\frac{1}{r^2(r+1)}$$

***B1**

Thus

$$\sum_{r=1}^{N}\frac{1}{r^2(r+1)}=\sum_{r=1}^{N}\frac{1}{r+1}-\sum_{r=1}^{N}\frac{1}{r}+\sum_{r=1}^{N}\frac{1}{r^2}$$

$$=\sum_{r=2}^{N+1}\frac{1}{r}-\sum_{r=1}^{N}\frac{1}{r}+\sum_{r=1}^{N}\frac{1}{r^2}$$

M1

$$=\frac{1}{N+1}-1+\sum_{r=1}^{N}\frac{1}{r^2}$$

as required. ***A1**

So, as $N\to\infty$,

$$LHS\to\sum_{r=1}^{\infty}\frac{1}{r^2(r+1)}\,,\quad \frac{1}{N+1}\to 0\text{ , and }\sum_{r=1}^{N}\frac{1}{r^2}\to\frac{1}{6}\pi^2$$

and hence

$$\sum_{r=1}^{\infty}\frac{1}{r^2(r+1)}=\frac{1}{6}\pi^2-1$$

***B1 (4)**

(ii)

$$\frac{1}{r^2(r+1)(r+2)}=\frac{A}{r}+\frac{B}{r^2}+\frac{C}{r+1}+\frac{D}{r+2}$$

M1

$$1=Ar(r+1)(r+2)+B(r+1)(r+2)+Cr^2(r+2)+Dr^2(r+1)$$

$$r=0\quad 1=2B\quad B=\frac{1}{2}$$

$$r=-1\quad 1=C$$

$$r=-2\quad 1=-4D\quad D=-\frac{1}{4}$$

$$r^3\text{ terms}\quad 0=A+C+D\quad A=-\frac{3}{4}$$

M1 A1 (3)

Thus

$$\sum_{r=1}^{N}\frac{1}{r^2(r+1)(r+2)}=-\frac{3}{4}\sum_{r=1}^{N}\frac{1}{r}+\frac{1}{2}\sum_{r=1}^{N}\frac{1}{r^2}+\sum_{r=1}^{N}\frac{1}{r+1}-\frac{1}{4}\sum_{r=1}^{N}\frac{1}{r+2}$$

$$=-\frac{3}{4}\sum_{r=1}^{N}\frac{1}{r}+\frac{1}{2}S_N+\sum_{r=2}^{N+1}\frac{1}{r}-\frac{1}{4}\sum_{r=3}^{N+2}\frac{1}{r}$$

M1

$$=-\frac{3}{4}-\frac{3}{8}+\frac{1}{2}S_N+\frac{1}{2}+\frac{1}{N+1}-\frac{1}{4}\frac{1}{N+1}-\frac{1}{4}\frac{1}{N+2}$$

dM1 A1ft

So

$$\sum_{r=1}^{N}\frac{1}{r^2(r+1)(r+2)}=\frac{1}{2}S_N+\frac{3}{4}\frac{1}{N+1}-\frac{1}{4}\frac{1}{N+2}-\frac{5}{8}$$

and taking limits as $N\to\infty$

M1

$$\sum_{r=1}^{\infty}\frac{1}{r^2(r+1)(r+2)}=\frac{1}{2}\times\frac{1}{6}\pi^2-\frac{5}{8}=\frac{1}{12}\pi^2-\frac{5}{8}$$

A1 (5)

(iii)

$$\frac{1}{r^2(r+1)^2}=\frac{A}{r}+\frac{B}{r^2}+\frac{C}{r+1}+\frac{D}{(r+1)^2}$$

M1

$$1=Ar(r+1)^2+B(r+1)^2+Cr^2(r+1)+Dr^2$$

$$r=0\quad 1=B$$

$$r=-1\quad 1=D$$

$$r^3\ terms\quad 0=A+C$$

$$r^2\ terms\quad 0=2A+B+C+D$$

M1

$$-2=2A+C$$

$$A=-2\quad C=2$$

Thus

$$\frac{1}{r^2(r+1)^2}=\frac{-2}{r}+\frac{1}{r^2}+\frac{2}{r+1}+\frac{1}{(r+1)^2}$$

A1 (3)

$$\sum_{r=1}^{N} \frac{1}{r^2(r+1)^2} = \sum_{r=1}^{N} \frac{-2}{r} + \sum_{r=1}^{N} \frac{1}{r^2} + \sum_{r=1}^{N} \frac{2}{(r+1)} + \sum_{r=1}^{N} \frac{1}{(r+1)^2}$$

$$= \sum_{r=1}^{N} \frac{-2}{r} + \sum_{r=1}^{N} \frac{1}{r^2} + \sum_{r=2}^{N+1} \frac{2}{\mathrm{r}} + \sum_{r=2}^{N+1} \frac{1}{r^2}$$

M1

$$= -2 + \frac{2}{N+1} + 2\sum_{r=1}^{N} \frac{1}{r^2} - 1 + \frac{1}{(N+1)^2}$$

That is

$$\sum_{r=1}^{N} \frac{1}{r^2(r+1)^2} = 2S_N - 3 + \frac{2}{N+1} + \frac{1}{(N+1)^2}$$

A1

From part (i),

$$\sum_{r=1}^{N} \frac{1}{r^2(r+1)} = \frac{1}{N+1} - 1 + \sum_{r=1}^{N} \frac{1}{r^2} = S_N - 1 + \frac{1}{N+1}$$

Thus

$$\sum_{r=1}^{N} \frac{1}{r^2(r+1)^2} = 2\left(\sum_{r=1}^{N} \frac{1}{r^2(r+1)} + 1 - \frac{1}{N+1}\right) - 3 + \frac{2}{N+1} + \frac{1}{(N+1)^2}$$

M1

$$= 2\sum_{r=1}^{N} \frac{1}{r^2(r+1)} - 1 + \frac{1}{(N+1)^2}$$

A1

Letting $N \to \infty$,

$$\sum_{r=1}^{\infty} \frac{1}{r^2(r+1)^2} = \sum_{r=1}^{\infty} \frac{2}{r^2(r+1)} - 1$$

B1* (5)

2. (i) (a)

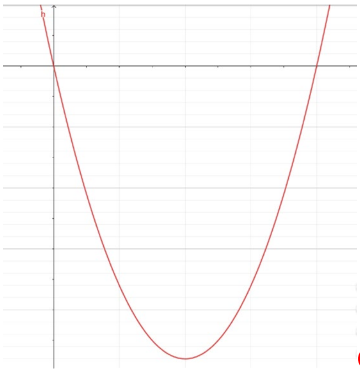
$$\sqrt{4x^2 - 8x + 64} \leq |x + 8|$$

$$4x^2 - 8x + 64 \leq (x + 8)^2 = x^2 + 16x + 64$$

Thus

$$3x^2 - 24x = 3x(x - 8) \leq 0$$

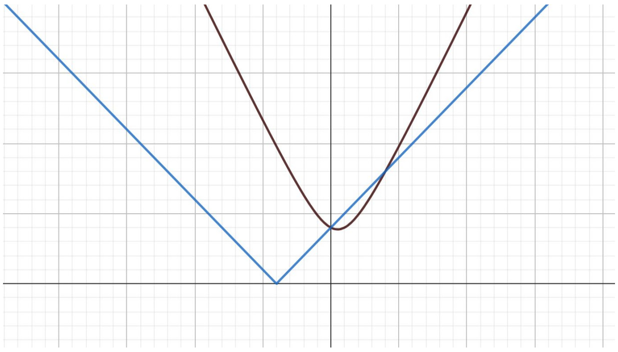
M1



G1 or consideration of intervals M1

[**Alternative method** Solve for critical values **M1**

Sketch graph



G1]

So $0 \leq x \leq 8$ **A1 (3)**

(b)

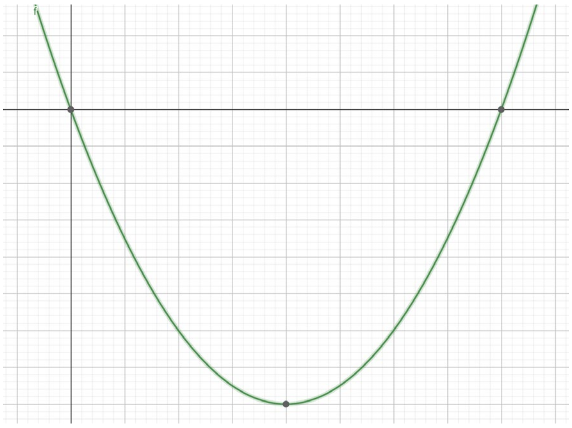
$$\sqrt{4x^2 - 8x + 64} \leq |3x - 8|$$

$$4x^2 - 8x + 64 \leq (3x - 8)^2 = 9x^2 - 48x + 64$$

Thus

$$5x^2 - 40x = 5x(x - 8) \geq 0$$

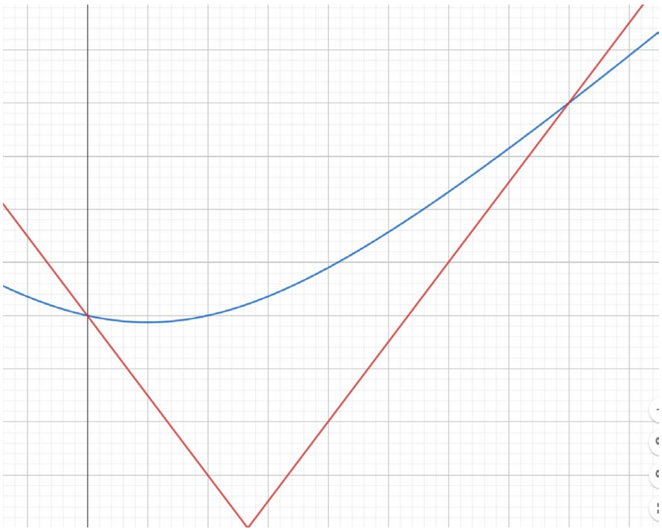
M1



GRAPH G1 or consideration of intervals M1

[Alternative method Solve for critical values M1

Sketch graph



G1]

So $x \leq 0$ or $x \geq 8$ A1 (3)

(ii) (a)

$$\left(\sqrt{4x^2 - 8x + 64} + 2(x - 1)\right) f(x)$$

$$= \left(\sqrt{4x^2 - 8x + 64} + 2(x - 1)\right)\left(\sqrt{4x^2 - 8x + 64} - 2(x - 1)\right)$$

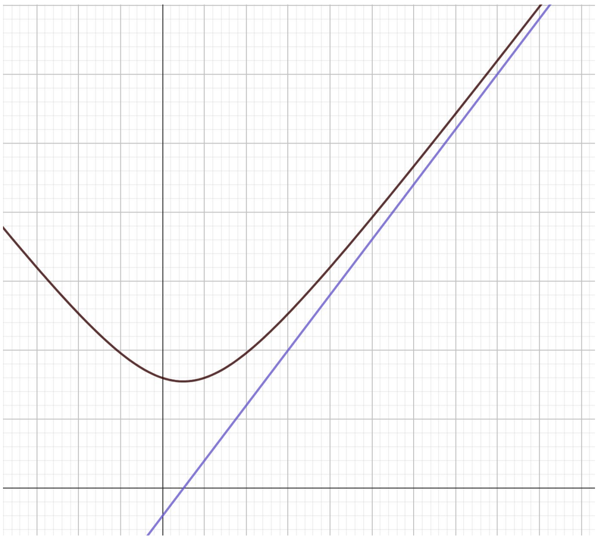
$$= 4x^2 - 8x + 64 - 4x^2 + 8x - 4 = 60$$

Thus $f(x) = \frac{60}{\left(\sqrt{4x^2-8x+64}+2(x-1)\right)}$

and so $f(x) \to 0$ as $x \to \infty$

E1 (1)

(b)



G2 (2)

(iii) Require one critical value 3, so $3m + c = \pm 5$

M1

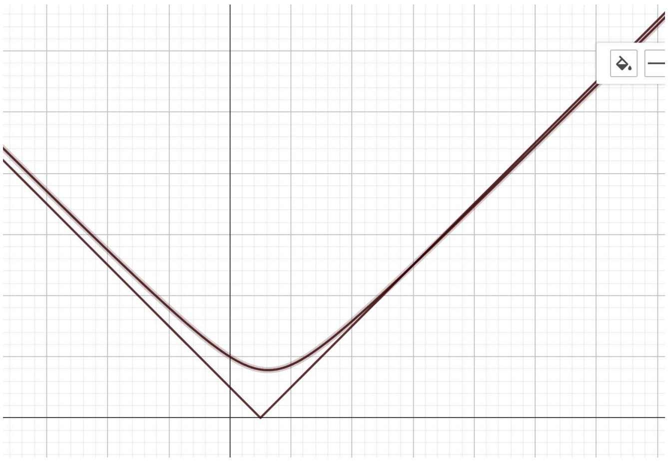
and as only one critical value choose $m = 2$ and $c = -1$, or $m = -2$ and $c = 1$

dM1 A1

$$\sqrt{4x^2 - 5x + 4} \leq |2x - 1|$$

$$4x^2 - 5x + 4 \leq (2x - 1)^2 = 4x^2 - 4x + 1$$

Giving $x \geq 3$ **M1**



G1 (5)

(iv) To obtain 4 critical values require quadratic to cross x axis and so

E1

$$x^2 + px + q = mx + c$$

has roots -5 and 7 giving $p - m = -2$ and $q - c = -35$

M1 A1

and

$$-(x^2 + px + q) = mx + c$$

has roots 1 and 5 giving $p + m = -6$ and $q + c = 5$

B1

Thus, $p = -4$, $q = -15$, $m = -2$, $c = 20$

A1

G1 (6)

3. (i) (a)

$$y = g(x) = \ln\left(1 + \frac{1}{x}\right) - \frac{x+c}{x(x+1)} = \ln(x+1) - \ln x - \frac{x+c}{x(x+1)}$$

$$\frac{dy}{dx} = \frac{1}{x+1} - \frac{1}{x} - \frac{x(x+1) - (x+c)(2x+1)}{x^2(x+1)^2}$$

M1

$$= \frac{x^2(x+1) - x(x+1)^2 - x(x+1) + (x+c)(2x+1)}{x^2(x+1)^2}$$

$$= \frac{(2c-1)x + c}{x^2(x+1)^2}$$

A1

When $c \geq \frac{1}{2}$, as $x > 0$, $(2c-1)x \geq 0$, and $c > 0$, so the numerator is positive and the denominator is a non-zero square so also positive, so $y = g(x)$ has positive gradient.

E1 (3)

(b) $y = g(x)$ has negative gradient for $0 \leq c < \frac{1}{2}$, if

$$(2c-1)x + c < 0$$

That is

$$(2c-1)x < -c$$

So

$$x > \frac{-c}{2c-1} = \frac{c}{1-2c}$$

B1 (1)

(ii) (a)

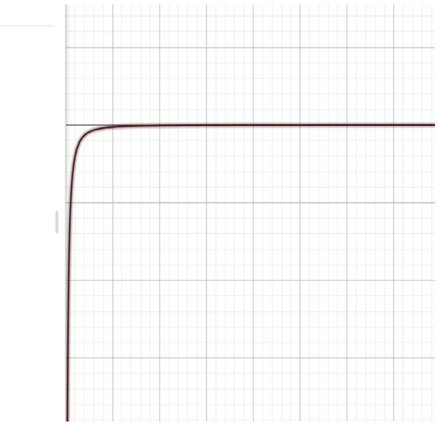
If $c = \frac{3}{4}$, then from (i) (a) the gradient is positive

E1

and we are given that $g(x) \to -\infty$ as $x \to 0$.

The gradient tends to zero as $x \to \infty$, and to ∞ as $x \to 0$. **E1**

Also, $g(x) \to 0$ as $x \to \infty$. **E1**



G1 (4)

(b)

If $c = \frac{1}{4}$, then from (i)(b) the gradient is negative for $x > \frac{1}{2}$.

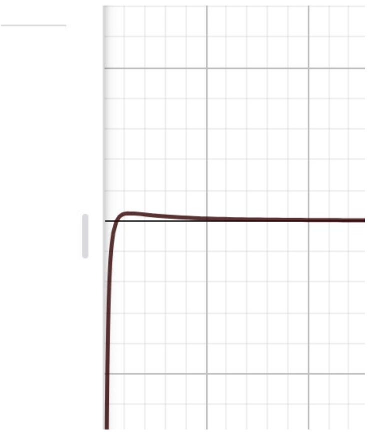
B1

The gradient is zero when $x = \frac{1}{2}$, and positive when $x < \frac{1}{2}$, and tending to zero as $x \to \infty$, and to ∞ as $x \to 0.$ **B1**

Again, we are given that $g(x) \to -\infty$ as $x \to 0$, and $g(x) \to 0$ as $x \to \infty$.

There is a turning point (maximum) at $\left(\frac{1}{2}, \ln 3 - 1\right) = \left(\frac{1}{2}, \ln \frac{3}{2}\right)$ which is above the x-axis.

M1 A1



G1 (5)

(iii)

$$f(x) = \left(1 + \frac{1}{x}\right)^{x+c}$$

$$\ln\big(f(x)\big) = (x + c)\ln\left(1 + \frac{1}{x}\right)$$

Thus

$$\frac{f'(x)}{f(x)} = \ln\left(1 + \frac{1}{x}\right) + (x + c)\left(\frac{1}{x + 1} - \frac{1}{x}\right) = g(x)$$

$$f'(x) = f(x)g(x)$$

M1 A1 (2)

Also, $f(x)$ is positive for $x > 0$.

(a) As has been demonstrated in (i) (a) and (ii) (a), $g(x) < 0$ for $x > 0$ when $c \geq \frac{1}{2}$, so $f'(x) < 0$ and f is a decreasing function. **E1**

(b) As has been demonstrated in (i) (b) and (ii) (b), $g(x) = 0$ for some $x > 0$ when $0 < c < \frac{1}{2}$, so $f'(x) = 0$ for some x and f has a turning point. **E1**

(c) When $c = 0$,

$$g'(x) = \frac{-1}{x(x + 1)^2}$$

Is always negative and $\rightarrow -\infty$ as $x \rightarrow 0$, and $\rightarrow 0$ as $x \rightarrow \infty$

E1

whilst $g(x) \rightarrow \infty$ as $x \rightarrow 0$, and $g(x) \rightarrow 0$ as $x \rightarrow \infty$

so $g(x)$ is positive for all $x > 0$, **E1**

thus $f'(x)$ is too and thus f is an increasing function

for all $x > 0$ **E1 (5)**

4. (i)

Suppose $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$, where $-\frac{1}{2}\pi < \theta_1, \theta_2 \leq \frac{1}{2}\pi$, then as the angle between the lines is 45^o, $\theta_1 - \theta_2 = \pm\frac{1}{4}\pi, or\ \pm\frac{3}{4}\pi$. **M1**

Therefore

$$\tan(\theta_1 - \theta_2) = \pm 1$$

and so

$$\tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2} = \pm 1$$

M1

i.e.

$$\frac{m_1 - m_2}{1 + m_1 m_2} = \pm 1$$

***A1 (3)**

(ii)

$$4ay = x^2$$

$$4a\frac{dy}{dx} = 2x$$

So the tangent at the point with x-coordinate p is

$$y - \frac{p^2}{4a} = \frac{p}{2a}(x - p)$$

$$4ay + p^2 = 2px$$

M1

The tangents $4ay + p^2 = 2px$, $4ay + q^2 = 2qx$ meet when

$2(p-q)x = p^2 - q^2 = (p-q)(p+q)$ and as $(p-q) \neq 0$,

M1

$$x = \frac{1}{2}(p+q)$$

***A1**

So

$$y = \frac{2p \times \frac{1}{2}(p+q) - p^2}{4a} = \frac{pq}{4a}$$

A1 (4)

and if the tangents meet at 45^o, then

$$\frac{\frac{p}{2a}-\frac{q}{2a}}{1+\frac{p}{2a}\frac{q}{2a}}=\pm 1$$

M1 M1

$$2a(p-q)=\pm(4a^2+pq)$$

$$(4a^2+pq)^2=4a^2(p-q)^2=4a^2((p+q)^2-4pq)$$

M1

Thus the point of intersection satisfies

$$(4a^2+4ay)^2=4a^2((2x)^2-16ay)$$

M1

That simplifies to

$$(a+y)^2=x^2-4ay$$

$$y^2+6ay+a^2=x^2$$

$$y^2+6ay+9a^2=x^2+8a^2$$

$$(y+3a)^2=x^2+8a^2$$

M1 *A1 (6)

(iii)

If

$$(y+7a)^2=48a^2+3x^2$$

$$\left(\frac{pq}{4a}+7a\right)^2=48a^2+3\left(\frac{1}{2}(p+q)\right)^2$$

M1

$$(pq+28a^2)^2=768a^4+12a^2(p+q)^2$$

$$p^2q^2+56a^2pq+784a^4=768a^4+12a^2(p-q)^2+48a^2pq$$

M1

$$p^2q^2+8a^2pq+16a^4=12a^2(p-q)^2$$

M1 A1

$$(pq+4a^2)^2=3\big(2a(p-q)\big)^2$$

$$\frac{\frac{p}{2a}-\frac{q}{2a}}{1+\frac{p}{2a}\frac{q}{2a}}=\pm\frac{1}{\sqrt{3}}$$

M1 A1

Thus the tangents are at a constant angle to each other which is 30^o. **A1 (7)**

5. (i)

Let

$$N=\begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$MN=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}=\begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

$$NM=\begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} ea+fc & eb+fd \\ ga+hc & gb+hd \end{pmatrix}$$

$$tr(MN)=ae+bg+cf+dh=ea+gb+fc+hd$$

$$=ea+fc+gb+hd=tr(NM)$$

M1A1

$$tr(M+N)=tr\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}+\begin{pmatrix} e & f \\ g & h \end{pmatrix}\right)=tr\begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}=a+e+d+h$$

$$=a+d+e+h=tr(M)+tr(N)$$

B1 (3)

(ii)

$$\frac{1}{\det M}\frac{d}{dt}(\det M)=\frac{1}{ad-bc}\times\frac{d}{dt}(ad-bc)=\frac{1}{ad-bc}\times\left(a\dot{d}+\dot{a}d-b\dot{c}-\dot{b}c\right)$$

M1

$$tr\left(M^{-1}\frac{dM}{dt}\right)=tr\left(\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} \dot{a} & \dot{b} \\ \dot{c} & \dot{d} \end{pmatrix}\right)=tr\left(\frac{1}{ad-bc}\begin{pmatrix} d\dot{a}-b\dot{c} & d\dot{b}-b\dot{d} \\ -c\dot{a}+a\dot{c} & -c\dot{b}+a\dot{d} \end{pmatrix}\right)$$

$$=\frac{1}{ad-bc}\times\left(d\dot{a}-b\dot{c}-c\dot{b}+a\dot{d}\right)=\frac{1}{\det M}\frac{d}{dt}(\det M)$$

M1 ***A1 (3)**

(iii)

$$tr\left(M^{-1}\frac{dM}{dt}\right)=tr\big(M^{-1}(MN-NM)\big)=tr(M^{-1}MN-M^{-1}NM)=tr(N)-tr(NMM^{-1})=0$$

Thus

$$\frac{1}{\det M}\frac{d}{dt}(\det M)=0$$

and so $\det M$ is independent of t

E1

$$\frac{d}{dt}\big(tr(M)\big)=tr\left(\frac{dM}{dt}\right)=tr(MN-NM)=tr(MN)-tr(NM)=0$$

so $tr(M)$ is independent of t

E1

$$tr(M^2) = tr\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = tr\begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} = a^2+bc+bc+d^2$$

$$= a^2+2ad+d^2+2(bc-ad) = (a+d)^2 - 2\,detM = (trM)^2 - 2detM$$

M1 A1

Therefore

$$\frac{d}{dt}\big(tr(M^2)\big) = \frac{d}{dt}\big((trM)^2\big) - \frac{d}{dt}(2detM) = 0$$

and so $tr(M^2)$ is independent of t

E1 (5)

$$\frac{dM}{dt} = MN - NM$$

So

$$\begin{pmatrix} \dot{A} & \dot{B} \\ \dot{C} & \dot{D} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} t & t \\ 0 & t \end{pmatrix} - \begin{pmatrix} t & t \\ 0 & t \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -Ct & At-Dt \\ 0 & Ct \end{pmatrix}$$

M1 A1

Thus C is a constant, **A1**

$A = a - \frac{1}{2}Ct^2, D = d + \frac{1}{2}Ct^2$ **A1**

and as

$$\dot{B} = At - Dt = (a-d)t - Ct^3$$

M1

$$B = b + \frac{1}{2}(a-d)t^2 - \frac{1}{4}Ct^4$$

A1 (6)

(iv) If

$$\frac{dM}{dt} = MN$$

Then for example, $M = \begin{pmatrix} e^t & 1+e^t \\ e^t & 1-e^t \end{pmatrix}$, $N = \begin{pmatrix} 1 & -e^{-t} \\ 0 & 1 \end{pmatrix}$

$$\frac{dM}{dt} = \begin{pmatrix} e^t & e^t \\ e^t & -e^t \end{pmatrix}$$

And $N = \begin{pmatrix} e^t & 1+e^t \\ e^t & 1-e^t \end{pmatrix}\begin{pmatrix} 1 & -e^{-t} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^t & e^t \\ e^t & -e^t \end{pmatrix}$

M1 A1

and

$$tr(N) = 2$$

so no. **A1 (3)**

6. (i) (a)

$$\frac{dx}{dt} = -x + 3y + u$$

$$\frac{dy}{dt} = x + y + u$$

$$\frac{dx}{dt} - \frac{dy}{dt} = \frac{d(x-y)}{dt} = -2(x-y)$$

M1

$$x - y = Ae^{-2t}$$

A1

If $x = y = 0$ at some time $t > 0$, then $A = 0$, **A1**

so considering $t = 0$, $x_0 - y_0 = 0$ which gives the required result. **E1 (4)**

(b) If $x_0 = y_0$, then at $t = 0$, $x - y = 0$ so $A = 0$ and hence $x = y$ for all t

E1

Thus

$$\frac{dx}{dt} = 2x + u$$

$$\frac{dx}{dt} - 2x = u$$

$$e^{-2t}\frac{dx}{dt} - 2e^{-2t}x = e^{-2t}u$$

$$e^{-2t}x = -\frac{1}{2}e^{-2t}u + c$$

$$x = -\frac{1}{2}u + ce^{2t}$$

M1 A1

$t = 0$, $x = x_0$ so $x_0 = -\frac{1}{2}u + c$ and we want $x = 0$ when $t = T$

so

$$0 = -\frac{1}{2}u + ce^{2T}$$

Thus $c = \frac{1}{2}ue^{-2T}$, $x_0 = -\frac{1}{2}u + \frac{1}{2}ue^{-2T}$

and hence,

$$u = \frac{2x_0e^{2T}}{1 - e^{2T}}$$

dM1 A1 (5)

(ii) (a)

$$\frac{dx}{dt} - 2\frac{dy}{dt} + \frac{dz}{dt} = \frac{d(x - 2y + z)}{dt} = -(x - 2y + z)$$

Thus

$$x - 2y + z = Ae^{-t}$$

M1 A1

If $x = y = z = 0$ at some time $t > 0$, then $A = 0$, so considering $t = 0$, $x_0 - 2y_0 + z_0 = 0$ which gives the required result. **E1 (3)**

(b) we know from (a) that if $x = y = z = 0$ at some time $t > 0$, then $A = 0$, and so

$x - 2y + z = 0$ or $2y = x + z$ **E1**

Thus

$$\frac{dx}{dt} = 2x - 3z + u$$

and

$$\frac{dz}{dt} = -z + u$$

So

$$\frac{dx}{dt} - \frac{dz}{dt} = \frac{d}{dt}(x - z) = 2(x - z)$$

and so

$$x - z = Be^{2t}$$

M1 A1

But as $x = z = 0$ at some time $t > 0$, $B = 0$ and so $x = z$ for all t

and thus $x = y = z$ for all t

Hence

$$x_0 = y_0 = z_0$$

E1 (4)

(c)

Given

$$x_0 = y_0 = z_0$$

we know that (a) and (b) apply (as similarly in (i)), so

$$\frac{dz}{dt} = -z + u$$

M1

Thus

$$z = u + ce^{-t}$$

A1

$t = 0$, $z = z_0$ so $z_0 = u + c$ and $0 = u + ce^{-T}$

dM1

$$c = -ue^{T}$$

$$u = \frac{z_0}{1 - e^{T}}$$

A1 (4)

7. (i) Each term of $f(n) > 0$ so their sum is too.

E1

$\frac{1}{(n+1)(n+2)\ldots(n+r)} < \frac{1}{(n+1)^r}$ so $f(n) = \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \cdots < \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots = \frac{1/n+1}{1-{}^1/_{n+1}} = \frac{1}{n}$

M1 A1(3)

Thus $0 < f(n) < \frac{1}{n}$

(ii) $\frac{1}{n+1} - \frac{1}{(n+1)(n+2)} > 0$, $\frac{1}{(n+1)(n+2)(n+3)} - \frac{1}{(n+1)(n+2)(n+3)(n+4)} > 0$, etc so $g(n) > 0$

M1 A1

Also, $\frac{1}{(n+1)(n+2)} - \frac{1}{(n+1)(n+2)(n+3)} > 0$, $\frac{1}{(n+1)(n+2)(n+3)(n+4)} - \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)} > 0$, etc

so $g(n) = \frac{1}{n+1}$ −a sum of positive terms $< \frac{1}{n+1}$ **M1 A1 (4)**

Thus $0 < g(n) < \frac{1}{n+1}$

(iii)

$$(2n)!\,e - f(2n)$$

$$= (2n)!\left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots\right) - \frac{1}{2n+1} - \frac{1}{(2n+1)(2n+2)} - \frac{1}{(2n+1)(2n+2)(2n+3)} - \cdots$$

M1 A1

$$= (2n)!\left(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{(2n)!}\right)$$

$$+ \frac{(2n)!}{(2n+1)!} + \frac{(2n)!}{(2n+2)!} + \cdots - \frac{1}{2n+1} - \frac{1}{(2n+1)(2n+2)} - \frac{1}{(2n+1)(2n+2)(2n+3)} - \cdots$$

$$= (2n)!\left(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{(2n)!}\right)$$

which is an integer. **M1 A1(4)**

$$\frac{(2n)!}{e} + g(2n) = (2n)!\,e^{-1} + g(2n)$$

$$= (2n)!\left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots\right) + \frac{1}{2n+1} - \frac{1}{(2n+1)(2n+2)} + \frac{1}{(2n+1)(2n+2)(2n+3)} - \cdots$$

M1 A1

$$= (2n)!\left(1 - 1 + \frac{1}{2!} + \cdots + \frac{1}{(2n)!}\right)$$

$$-\frac{(2n)!}{(2n+1)!} + \frac{(2n)!}{(2n+2)!} - \cdots + \frac{1}{2n+1} - \frac{1}{(2n+1)(2n+2)} + \frac{1}{(2n+1)(2n+2)(2n+3)} - \cdots$$

$$= (2n)!\left(1 - 1 + \frac{1}{2!} + \cdots + \frac{1}{(2n)!}\right)$$

M1 A1 (4)

which is an integer.

(iv) $q\big((2n)!\,e - f(2n)\big)$ is an integer as is $p\left(\frac{(2n)!}{e} + g(2n)\right)$

Thus $\left(p\left(\frac{(2n)!}{e} + g(2n)\right)\right) - \left(q\big((2n)!\,e - f(2n)\big)\right)$ is an integer.

$$\left(p\left(\frac{(2n)!}{e} + g(2n)\right)\right) - \left(q\big((2n)!\,e - f(2n)\big)\right) = (2n)!\left(\frac{p}{e} - qe\right) + pg(2n) + qf(2n)$$

M1

$= pg(2n) + qf(2n)$

so $pg(2n) + qf(2n)$ is an integer as required. **A1(2)**

(v) As (iv) is true for all positive integers n, it must be true for $n = \max(p, q)$

By (ii) $\quad pg(2n) < \frac{p}{2n+1} \leq \frac{n}{2n+1} < \frac{1}{2}$

By (i) $\quad qf(2n) < \frac{q}{2n} \leq \frac{n}{2n} = \frac{1}{2}$

M1

Therefore, $\quad pg(2n) + qf(2n) < \frac{1}{2} + \frac{1}{2} = 1$

and trivially by (i) and (ii)

$$pg(2n) + qf(2n) > 0$$

A1

This means that $pg(2n) + qf(2n)$ cannot be an integer which contradicts the result of (iv) and hence there are no integers such that $\frac{p}{e} = qe$, that is such that $\frac{p}{q} = e^2$ and so e^2 is irrational. **E1 (3)**

8. (i) $(y-x+3)(y+x-5)=0$ if and only if either $y-x+3=0$ or $y+x-5=0$. These are the equations of two straight lines with gradients 1 and -1.

E1

A pair of straight lines with gradients 1 and -1 can be expressed as $y-x+a=0$ and $y+x+b=0$. Thus $(y-x+a)(y+x+b)=0$ can be expressed

$y^2-x^2+py+qx+r=0$ if and only if $a+b=p$, $a-b=q$, and $ab=r$. **M1 A1**

Hence, $a=\frac{1}{2}(p+q)$, $b=\frac{1}{2}(p-q)$ and so $\frac{1}{2}(p+q)\frac{1}{2}(p-q)=r$ which can be written

$p^2-q^2=4r$. **M1 *A1 (5)**

(ii) If a point (x,y) lies on C_1, then $x=y^2+2sy+s(s+1)=0$ which can be rearranged as $y^2+2sy+s(s+1)-x=0$. If a point (x,y) lies on C_2, then $y=x^2$ which can be expressed as $k(y-x^2)=0$ for any real number k. Thus, if it lies on both

$y^2+2sy+s(s+1)-x+k(y-x^2)=0$ for any real number k

E1 E1

If $k=1$,

$$y^2-x^2+(2s+1)y-x+s(s+1)=0$$

and

$$(2s+1)^2-(-1)^2=4s^2+4s=4s(s+1)$$

satisfying the condition as derived in (i). **M1 A1 (4)**

(iii) If C_1 and C_2 intersect at four distinct points, then they do so on the pair of straight lines,

$y^2-x^2+(2s+1)y-x+s(s+1)=0$.

E1

$y^2-x^2+(2s+1)y-x+s(s+1)=(y+x+s+1)(y-x+s)$ **B1**

Therefore, $y+x+s+1=0$ must meet C_2 at two distinct points and $y-x+s=0$ must meet C_2 at two different distinct points. **E1**

Thus, solving $x^2+x+s+1=0$ having two distinct roots, the discriminant $1-4(s+1)>0$

That is $s<-\frac{3}{4}$ **M1 A1**

and solving $x^2-x+s=0$ having two distinct roots, the discriminant $1-4s>0$ i.e. $s<\frac{1}{4}$

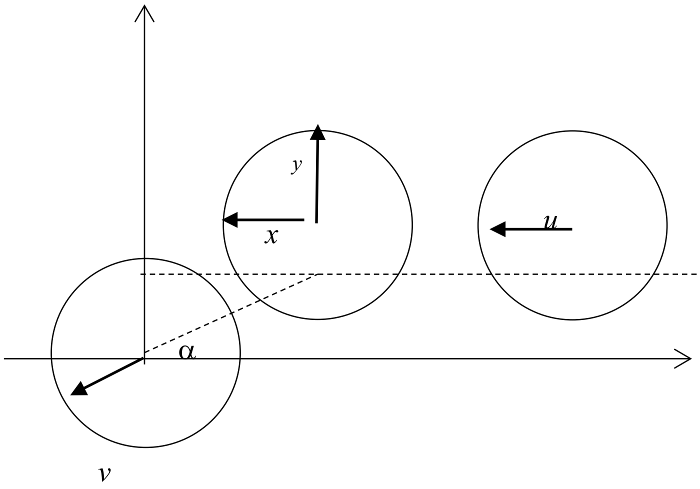
So it is necessary that $s<-\frac{3}{4}$. ***A1 (6)**

(iv) If $s < -\frac{3}{4}$ for C_1 and C_2 to intersect at four points, they do so on the pair of straight lines, two distinct on each of the lines in (iii) as shown by the non-zero discriminants in (iii) and that will be four distinct points provided that the point of intersection of the two lines is not one of them. **E1 E1**

The intersection of

$y + x + s + 1 = 0$ and $y - x + s = 0$ is at $\left(-\frac{1}{2}, -\frac{2s+1}{2}\right)$ which will only lie on C_2 if $-\frac{2s+1}{2} = \left(-\frac{1}{2}\right)^2$ that is if $s = -\frac{3}{4}$ which is prohibited. **M1 A1 E1 (5)**

9.



G1 (1)

(i) Conserving momentum in the direction of the line of centres

$$mu\cos\alpha = mx\cos\alpha - my\sin\alpha + mv$$

that is

$u\cos\alpha = (x\cos\alpha - y\sin\alpha) + v$ **M1**

Newton's experimental law of impact in the same direction gives

$$v - (x\cos\alpha - y\sin\alpha) = \frac{1}{3}u\cos\alpha$$

Solving, $(x\cos\alpha - y\sin\alpha) = \frac{1}{3}u\cos\alpha$ (A)

M1

Conserving momentum perpendicular to the direction of the line of centres

M1

$$mu\sin\alpha = m(x\sin\alpha + y\cos\alpha)$$

that is

$(x\sin\alpha + y\cos\alpha) = u\sin\alpha$ (B)

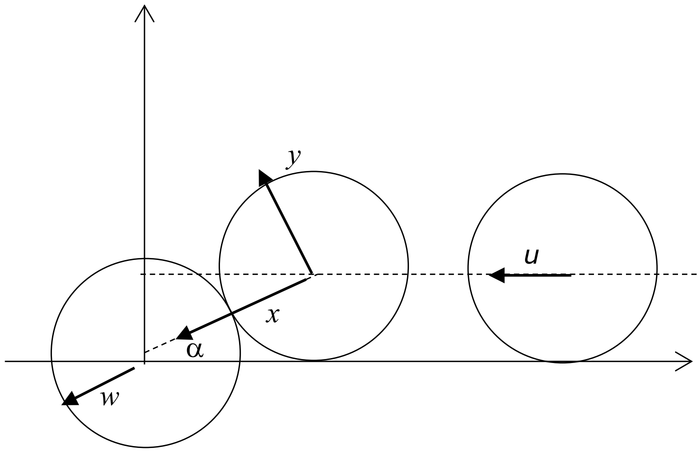
Solving equations A and B simultaneously:-

$A\cos\alpha + B\sin\alpha$ gives $x = u\left(\frac{1}{3}\cos^2\alpha + \sin^2\alpha\right) = \frac{1}{3}u(1 + 2\sin^2\alpha)$

$B\cos\alpha - A\sin\alpha$ gives $y = \frac{2}{3}u\sin\alpha\cos\alpha$ **M1**

Thus the velocity of B after the collision is $\begin{pmatrix} -\frac{1}{3}u(1 + 2\sin^2\alpha) \\ \frac{2}{3}u\sin\alpha\cos\alpha \end{pmatrix}$ as required. ***A1 (5)**

Alternative for part (i)



G1 (1)

Conserving momentum in the direction of the line of centres

$$mu\cos\alpha = mx + mw$$

That is $$u\cos\alpha = x + w$$

M1

Newton's experimental law of impact in the same direction gives

$$w - x = \frac{1}{3}u\cos\alpha$$

Solving, $x = \frac{1}{3}u\cos\alpha$

M1

Conserving momentum perpendicular to the direction of the line of centres

M1

$$mu\sin\alpha = my$$

that is

$$y = u\sin\alpha$$

Thus the velocity of B after the collision is

$$\begin{pmatrix} -\frac{1}{3}u\cos\alpha\cos\alpha - u\sin\alpha\sin\alpha \\ u\sin\alpha\cos\alpha - \frac{1}{3}u\cos\alpha\sin\alpha \end{pmatrix} = \begin{pmatrix} -\frac{1}{3}u(1 + 2\sin^2\alpha) \\ \frac{2}{3}u\sin\alpha\cos\alpha \end{pmatrix}$$ as required. **M1 *A1 (5)**

(ii) The lowest point of sphere B (which is vertically below its centre) crosses the y axis at a time

$$\frac{2r\cos\alpha}{\frac{1}{3}u(1+2\sin^2\alpha)}$$

after the collision. **B1**

Thus it crosses the y axis at a point

$$2r\sin\alpha+\frac{2}{3}u\sin\alpha\cos\alpha\frac{2r\cos\alpha}{\frac{1}{3}u(1+2\sin^2\alpha)}=2r\left(\sin\alpha+\cos\alpha\frac{2\sin\alpha\cos\alpha}{1+2\sin^2\alpha}\right)$$

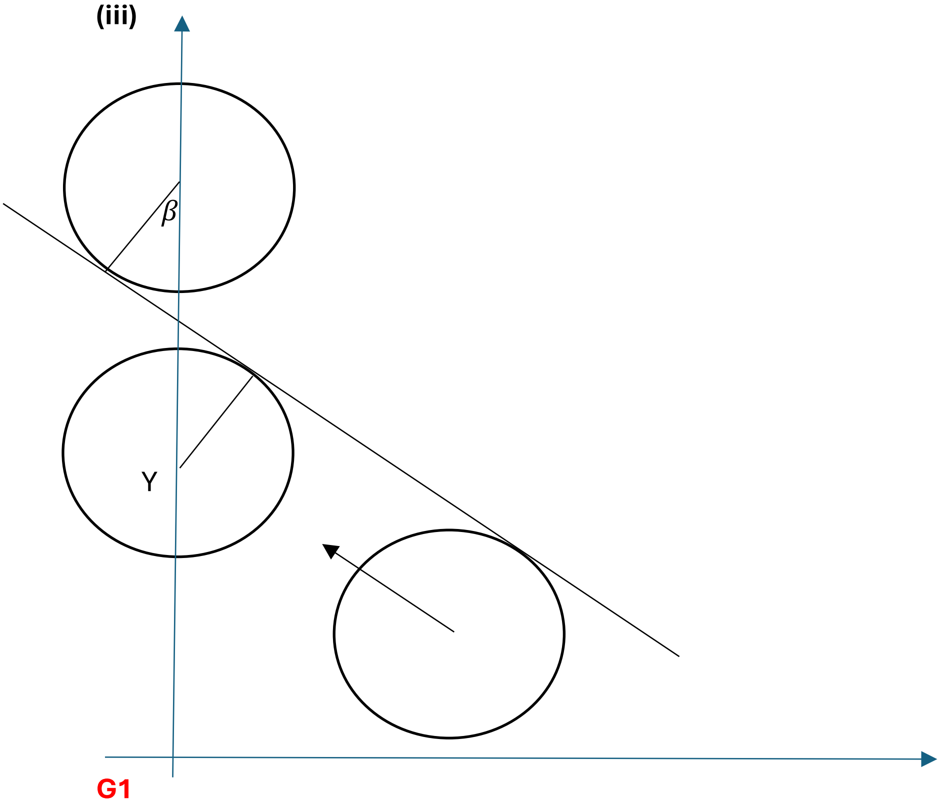
M1

from the origin, that is at $(0,Y)$ where $Y=2r(\sin\alpha+\cos\alpha\tan\beta)$ and $\tan\beta=\frac{2\sin\alpha\cos\alpha}{1+2\sin^2\alpha}$

***A1 (3)**

Alternatively, after the collision sphere B moves at an angle β above the negative x axis, where $\tan\beta=\frac{2\sin\alpha\cos\alpha}{1+2\sin^2\alpha}$ (from the velocity of B in (i)) and so by trigonometry, the lowest point will be a distance $2r\cos\alpha\tan\beta$ further in the y direction than the point it is at when the collision occurs which had a y coordinate $2r\sin\alpha$.

(iii)



G1

By trigonometry, the distance between the lowest points of spheres B and C when their lowest points are on the y axis must be more than $2r\sec\beta$ so to avoid any contact, $h>Y+2r\sec\beta$.

E1 (2)

(iv)

$$Y = 2r(\sin\alpha + \cos\alpha\tan\beta) = \frac{2r}{\cos\beta}(\sin\alpha\cos\beta + \cos\alpha\sin\beta) = 2r\sec\beta\ \sin(\alpha+\beta) \leq 2r\sec\beta$$

as $\sin(\alpha+\beta) \leq 1$ **M1 A1**

But, $\alpha + \beta \neq \frac{\pi}{2}$ because expression (A) in (i) cannot be zero, or alternatively, $\tan\beta = \frac{2\sin\alpha\cos\alpha}{1+2\sin^2\alpha}$ would give a contradiction. Thus $Y < 2r\sec\beta$ **E1 (3)**

So, from this result and (iii) there will be no striking if $h > 4r\sec\beta$ **B1**

The greatest value of $\sec\beta$ occurs when $\tan\beta$ is greatest. **E1**

$$\frac{d}{d\alpha}(\tan\beta) = \frac{d}{d\alpha}\left(\frac{2\sin\alpha\cos\alpha}{1+2\sin^2\alpha}\right) = \frac{(1+2\sin^2\alpha)2(\cos^2\alpha - \sin^2\alpha) - 2\sin\alpha\cos\alpha\, 4\sin\alpha\cos\alpha}{(1+2\sin^2\alpha)^2}$$

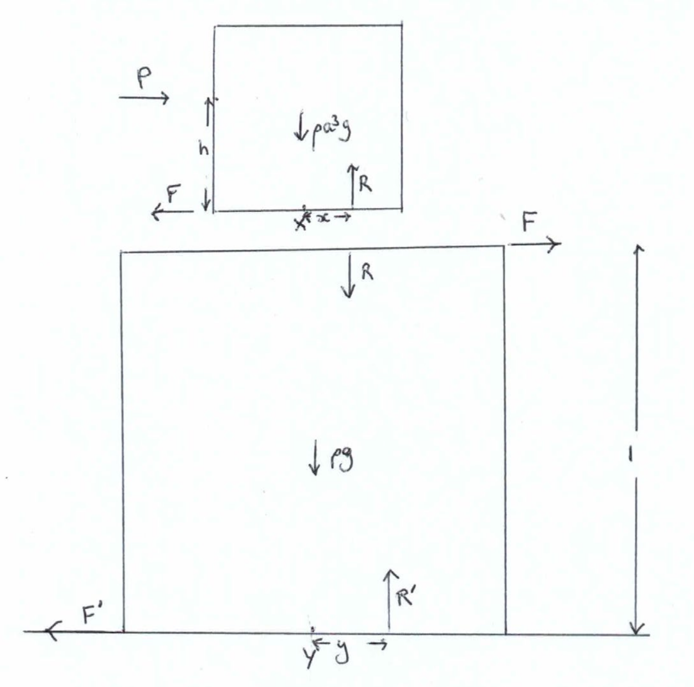
Numerator $= (1+2s^2)2(1-2s^2) - 8s^2(1-s^2) = 2 - 8s^2$ **M1 A1**

where $s = \sin\alpha$

Thus, the differential is zero when $\sin\alpha = \frac{1}{2}$, then $\tan\beta = \frac{1}{\sqrt{3}}$ and $\sec\beta = \frac{2}{\sqrt{3}}$ **M1**

Thus, no collision occurs for any value of α if $h > \frac{8r}{\sqrt{3}}$ ***A1 (6)**

10. (i)



G1

A Resolving vertically for the upper cube $R = \rho a^3 g$

B Taking moments for the upper cube about X $Ph = Rx$

C Resolving horizontally for the upper cube $P = F$

M1 A1

D Resolving vertically for the lower cube $R' = R + \rho g$

E Taking moments for the lower cube about Y $R'y = F + Rx$

F Resolving horizontally for the lower cube $F = F'$

M1 A1

Combining A and B $x = \frac{Ph}{\rho a^3 g}$ which is the second requirement. **B1**

Combining A and D (or resolving vertically for the whole system) $R' = \rho g(1 + a^3)$

Combining this with E and C $\rho g(1 + a^3)y = P + Ph$

which gives $y = \frac{P(1+h)}{(1+a^3)\rho g}$ as required. **M1 *A1 (8)**

(ii) The limiting friction for $F = \mu R$ whereas for $F' = \mu R' = \mu R + \mu \rho g > \mu R$

Given that in equilibrium $F = F' = P$, as P increases, F will attain its limiting value first and hence the upper cube will slip on the lower cube. **E1 (1)**

(iii) For $a = 1$ the upper cube slips on the lower cube if $P = \mu\rho g$, **B1**

$y = \frac{1}{2}$ when $P = \frac{\rho g}{1+h}$

and $x = \frac{1}{2}$ when $P = \frac{\rho g}{2h}$ and as $h < 1$, $2h < 1 + h$, so $\frac{\rho g}{2h} > \frac{\rho g}{1+h}$ **M1**

Thus equilibrium is broken either by the upper cube slipping if $\mu\rho g < \frac{\rho g}{1+h}$ or by both toppling together if $\mu\rho g > \frac{\rho g}{1+h}$ **M1**

That is upper slips if $\mu(1 + h) < 1$ or both topple if $\mu(1 + h) > 1$ as required. ***A1 (4)**

(iv) For $a < 1$ and no slipping occurring,

$y = \frac{1}{2}$ when $P = \frac{\rho g(1+a^3)}{2(1+h)}$ **B1**

and $x = \frac{1}{2}a$ when $P = \frac{\rho g a^4}{2h}$ **B1**

So equilibrium will be broken by the upper cube toppling if $\frac{\rho g a^4}{2h} < \frac{\rho g(1+a^3)}{2(1+h)}$

That is if $(1 + h)a^4 < h(1 + a^3)$, which can be rearranged to $h\big(1 + a^3(1 - a)\big) > a^4$ **E1 (3)**

(v) If $a = \frac{1}{2}$, for (iv) to occur $h > \frac{1}{17}$. **B1**

We also require $h < a$ and, in addition, $\frac{\rho g a^4}{2h} < \mu\rho g a^3$ so that the top cube doesn't slip. **M1**

Thus, $h\mu > \frac{1}{4}$. **A1**

E.G. $h = \frac{3}{8}$, $\mu = \frac{3}{4}$ (choosing h larger than $\frac{1}{4}$ to enable a feasible value of μ to be chosen.)

B1 (4)

11. (i)

$$r\binom{2n}{r} = \frac{(2n)!}{(2n-r)!\,(r-1)!} = (2n-r+1)\frac{(2n)!}{(2n-r+1)!\,(r-1)!} = (2n-r+1)\binom{2n}{2n+1-r}$$

M1 **M1** ***A1 (3)**

So

$$\sum_{r=0}^{2n} r\binom{2n}{r} = \sum_{r=1}^{n} r\binom{2n}{r} + \sum_{r=n+1}^{2n} r\binom{2n}{r}$$

M1

$$= \sum_{r=1}^{n} (2n-r+1)\binom{2n}{2n+1-r} + \sum_{r=n+1}^{2n} r\binom{2n}{r}$$

M1

$$= \sum_{t=n+1}^{2n} t\binom{2n}{t} + \sum_{r=n+1}^{2n} r\binom{2n}{r}$$

by changing the index in the first summation to $t = 2n - r + 1$ **M1 A1**

$$= 2\sum_{r=n+1}^{2n} r\binom{2n}{r}$$

***A1 (5)**

(ii)

$$E(X) = 2\sum_{r=n+1}^{2n} r\binom{2n}{r}\left(\frac{1}{2}\right)^{2n} + n\binom{2n}{n}\left(\frac{1}{2}\right)^{2n}$$

M1 A1

$$= \sum_{r=0}^{2n} r\binom{2n}{r}\left(\frac{1}{2}\right)^{2n} + n\binom{2n}{n}\left(\frac{1}{2}\right)^{2n}$$

using the result of (i) **M1 A1**

But,

$$\sum_{r=0}^{2n} r\binom{2n}{r}\left(\frac{1}{2}\right)^{2n}$$

is the expectation of a $Bi\left(2n, \frac{1}{2}\right)$ random variable and so $= 2n \times \frac{1}{2} = n$, or using the second result given in the stem **M1 A1**

Thus $E(X) = n + n\binom{2n}{n}\left(\frac{1}{2}\right)^{2n} = n\left(1 + \left(\frac{1}{2}\right)^{2n}\binom{2n}{n}\right)$

as required. ***A1 (7)**

(iii)

$$\frac{\left(\frac{1}{2}\right)^{2n+2}\binom{2n+2}{n+1}}{\left(\frac{1}{2}\right)^{2n}\binom{2n}{n}} = \frac{1}{2^2}\frac{(2n+2)!}{(n+1)!\,(n+1)!}\frac{n!\,n!}{(2n)!} = \frac{(2n+2)(2n+1)}{2^2(n+1)(n+1)} = \frac{2n+1}{2(n+1)} < 1$$

M1 **A1**

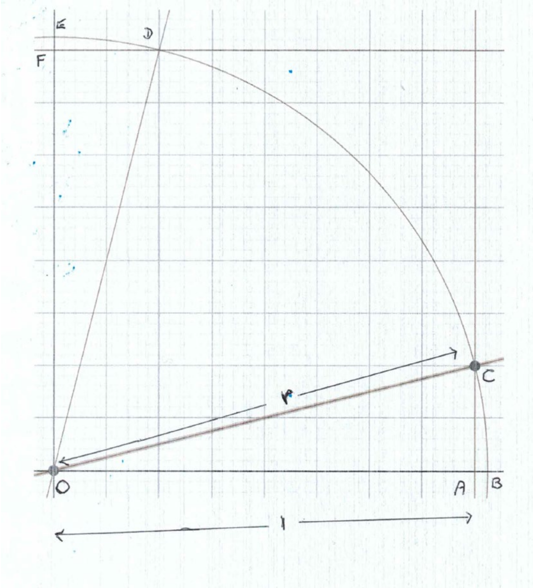
and so $\left(\frac{1}{2}\right)^{2n}\binom{2n}{n}$ decreases as n increases. **E1 (3)**

(iv) The expected profit per pound paid $= \dfrac{n\left(1+\left(\frac{1}{2}\right)^{2n}\binom{2n}{n}\right)-n}{n} = \left(\frac{1}{2}\right)^{2n}\binom{2n}{n}$ **M1**

So, by the result of (iii), choose $n = 1$. **A1 (2)**

(If winnings not profit, then logic is identical bar plus 1. Either permissible)

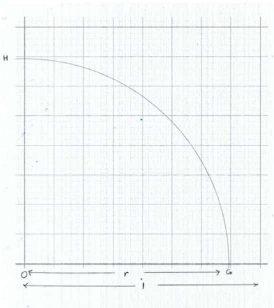
12. (i)



$OA = 1$, $OC = r$, so $AC = \sqrt{r^2 - 1}$ and therefore $area\ OAC = ODF = \frac{1}{2}\sqrt{r^2 - 1}$

Also, $angle\ BOC = DOE = \cos^{-1}\frac{1}{r}$ so $area\ sector\ COD = \frac{1}{4}\pi r^2 - 2 \times \frac{1}{2} r^2 \cos^{-1}\frac{1}{r}$ **M1 M1**

Thus, $P(R \leq r) = area\ OACDF = \sqrt{r^2 - 1} + \frac{1}{4}\pi r^2 - r^2 \cos^{-1} r^{-1}$ when $1 \leq r \leq \sqrt{2}$ ***A1**



$P(R \leq r) = area\ OGH = \frac{1}{4}\pi r^2$ when $0 \leq r \leq 1$ **B1 (4)**

(ii) If the pdf of R is $f(r)$, then differentiating,

$$f(r) = \frac{1}{2}\pi r$$

for $0 \leq r \leq 1$,

and using

$$\frac{d}{dx}(\cos^{-1} x^{-1}) = -\frac{1}{\sqrt{1 - x^{-2}}} \times -x^{-2} = \frac{1}{x\sqrt{x^2 - 1}}$$

B1

$$f(r) = \frac{r}{\sqrt{r^2 - 1}} + \frac{1}{2}\pi r - 2r\cos^{-1} r^{-1} - \frac{r}{\sqrt{r^2 - 1}} = \frac{1}{2}\pi r - 2r\cos^{-1} r^{-1}$$

for $1 \le r \le \sqrt{2}$. **M1 A1 (3)**

Thus

$$E(R) = \int_0^1 \frac{1}{2}\pi r^2 \, dr + \int_1^{\sqrt{2}} \frac{1}{2}\pi r^2 - 2r^2 \cos^{-1} r^{-1} \, dr$$

M1 A1ft

$$= \left[\frac{1}{6}\pi r^3\right]_0^{\sqrt{2}} - \left[\frac{2}{3} r^3 \cos^{-1} r^{-1}\right]_1^{\sqrt{2}} + \int_1^{\sqrt{2}} \frac{2}{3} r^3 \frac{1}{r\sqrt{r^2-1}} dr$$

M1 A1

$$= \frac{\pi\sqrt{2}}{3} - \frac{\pi\sqrt{2}}{3} + \int_1^{\sqrt{2}} \frac{2}{3} r^2 \frac{1}{\sqrt{r^2-1}} dr$$

as required. ***A1 (5)**

(iii)

Let

$$I = \int \frac{r^2}{\sqrt{r^2-1}} dr$$

Then

$$I = \int r \frac{r}{\sqrt{r^2-1}} dr = r\sqrt{r^2-1} - \int \sqrt{r^2-1} \, dr = r\sqrt{r^2-1} - \int \frac{r^2-1}{\sqrt{r^2-1}} dr$$

M1 **M1**

$$= r\sqrt{r^2-1} - \int \frac{r^2}{\sqrt{r^2-1}} dr + \int \frac{1}{\sqrt{r^2-1}} dr = r\sqrt{r^2-1} - I + \cosh^{-1} r + c$$

M1 **M1**

So

$$I = \frac{1}{2}\left(r\sqrt{r^2-1} + \cosh^{-1} r + c\right) = \frac{1}{2}\left(r\sqrt{r^2-1} + \ln\left(r + \sqrt{r^2-1}\right) + c\right)$$

A1 (5)

Hence

$$E(R) = \frac{2}{3} \times \frac{1}{2} \times \left[r\sqrt{r^2-1} + \ln\left(r + \sqrt{r^2-1}\right)\right]_1^{\sqrt{2}} = \frac{1}{3}\left(\sqrt{2} + \ln(\sqrt{2}+1)\right)$$

M1 A1ft ***A1 (3)**

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