

STEP II 2012 Q1

$$(1 - x^6)^{-2} = 1 + 2x^6 + 3x^{12} + 4x^{18} + 5x^{24} + \dots$$

General term $(n + 1)x^{6n}$

B1 Enough terms to see what's going on

B1 (Implies previous **B1**'s work) **2 marks**

(i) $(1 - x^3)^{-1} = (1 + x^3 + x^6 + x^9 + \dots)$ **B1**

The x^{24} term in $(1 - x^6)^{-2}(1 - x^3)^{-1} = (1 + 2x^6 + 3x^{12} + 4x^{18} + 5x^{24} + \dots)(1 + x^3 + x^6 + x^9 + \dots)$ comes from $1 \cdot x^{24} + 2x^6 \cdot x^{18} + 3x^{12} \cdot x^{12} + 4x^{18} \cdot x^6 + 5x^{24} \cdot 1$

M1

Coefft. of x^{24} is $1 + 2 + 3 + 4 + 5 = 15$ **A1**

3 marks

$$\text{Coefft. of } x^n \text{ is } \begin{cases} 0 & n = 6k + \{1, 2, 4, 5\} \\ \frac{1}{2}(k+1)(k+2) & n = 6k + 3 \\ \frac{1}{2}(k+1)(k+2) & n = 6k \end{cases}$$

B1 B1 B1

One each correct

3 marks

(ii) $f(x) = (1 + 2x^6 + 3x^{12} + 4x^{18} + 5x^{24} + \dots)(1 + x^3 + x^6 + x^9 + \dots)(1 + x + x^2 + x^3 + \dots)$

The x^{24} term comes from

$$\begin{aligned} & 1 \cdot 1 \cdot 5x^{24} + 1 \cdot x^6 \cdot 4x^{18} + 1 \cdot x^{12} \cdot 3x^{12} + 1 \cdot x^{18} \cdot 2x^6 + 1 \cdot x^{24} \cdot 1 \\ & + x^3 \cdot x^3 \cdot 4x^{18} + x^3 \cdot x^9 \cdot 3x^{12} + x^3 \cdot x^{15} \cdot 2x^6 + x^3 \cdot x^{21} \cdot 1 \\ & + x^6 \cdot 1 \cdot 4x^{18} + x^6 \cdot x^6 \cdot 3x^{12} + x^6 \cdot x^{12} \cdot 2x^6 + x^6 \cdot x^{18} \cdot 1 \\ & + x^9 \cdot x^3 \cdot 3x^{12} + x^9 \cdot x^9 \cdot 2x^6 + x^9 \cdot x^{15} \cdot 1 \\ & + x^{12} \cdot 1 \cdot 3x^{12} + x^{12} \cdot x^6 \cdot 2x^6 + x^{12} \cdot x^{12} \cdot 1 \\ & + x^{15} \cdot x^3 \cdot 2x^6 + x^{15} \cdot x^9 \cdot 1 \\ & + x^{18} \cdot 1 \cdot 2x^6 + x^{18} \cdot x^6 \cdot 1 \\ & + x^{21} \cdot x^3 \cdot 1 \\ & + x^{24} \cdot 1 \cdot 1 \end{aligned}$$

Coefft. of x^{24} is $15 + 2 \times (10 + 6 + 3 + 1) = 55$

M1 First **M** mark for keeping one term fixed from any bracket

M1 Second **M** mark for 2nd bracket

M1 Third **M** mark for fully correct method

A1 Answer Given

4 marks

Alternatively, the sum is simply

$$5 \times 1 + 4 \times 3 + 3 \times 5 + 2 \times 7 + 1 \times 9 = 5 + 12 + 15 + 14 + 9 = 55. \quad (*)$$

Note that, because every non-multiple-of-3 power in bracket 3 is redundant, the x^{24} term comes from considering $f(x) = (1 - x^6)^{-2}(1 - x^3)^{-2} = (1 + 2x^6 + 3x^{12} + 4x^{18} + 5x^{24} + \dots)(1 + 2x^3 + 3x^6 + 4x^9 + \dots)$.

Again, every non-multiple-of-6 power in *this* 2nd bracket is also redundant, one might consider only

$$f(x) = (1 + 3x^6 + 5x^{12} + 7x^{18} + 9x^{24} + \dots)(1 + 2x^6 + 3x^{12} + 4x^{18} + 5x^{24} + \dots)$$

from which the coefft. of x^{24} is simply calculated as $1 \times 5 + 3 \times 4 + 5 \times 3 + 7 \times 2 + 9 \times 1 = 55$, exactly as

in (*). The result (*), in some form or another – i.e. working from the 3rd bracket – gives the way of working out the coefficient of x^{6n} for any non-negative integer n . It is immediately obvious that it

$$\text{is } \sum_{r=0}^n (n+1-r)(2r+1) \text{ which turns out to be the same as } \sum_{r=1}^{n+1} r^2 = \frac{1}{6}(n+1)(n+2)(2n+3).$$

The proof of this result could be by induction or direct manipulation of the standard results for Σr and Σr^2 . However, I very much doubt any candidate will approach it in this general way and I am not presently requiring proofs of such results are not required.

The coefft. of x^{25} is 55		B1	
This is the same as for x^{24} , since the extra x only arises from replacing 1 by x , x^3 by x^4 , etc., in the first bracket's term (at each step of the working) and the coefficients are equal in each case.		E1	Credible reasoning (there must be some)
		Or	similar working repeated 2 marks
In the case when $n = 11$, the coefficient of x^{66} is gained from			
$12x^{66} \times$ the no. of ways of getting x^0 from the first two brackets		M1	
$+ 11x^{60} \times$ the no. of ways of getting x^6 from the first two brackets		M1	
$+ 10x^{54} \times$ the no. of ways of getting x^{12} from the first two brackets		M1	
$+ \dots$			
$+ 2x^6 \times$ the no. of ways of getting x^{60} from the first two brackets			
$+ 1x^0 \times$ the no. of ways of getting x^{66} from the first two brackets		M1	all the way down
$= 12 \times 1 + 11 \times 3 + 10 \times 5 + \dots + 2 \times 21 + 1 \times 23$		M1	
$= 12 + 33 + 50 + 63 + 72 + 77 + 78 + 75 + 68 + 57 + 42 + 23$			
$= 650$		A1	6 marks
STEP II 2012 Q2			
$p(q(x))$ has degree mn		B1	1 mark
(i)	$\text{Deg}[p(x)] = n \Rightarrow \text{Deg}[p(p(x))] = n^2 \ \& \ \text{Deg}[p(p(p(x)))] = n^3$ $\text{Deg}[\text{LHS}] \leq \max(n^3, n)$. RHS of degree 1. Therefore LHS not constant (nb $n \geq 0$) so $n = 1$ and $p(x)$ is linear.	B1	Noted somewhere
		E1	Essentially correct reasoning 2 marks
Setting $p(x) = ax + b$			
$\Rightarrow p(p(x)) = a(ax + b) + b = a^2x + (a + 1)b$		M1	
$\& \ p(p(p(x))) = a[a^2x + (a + 1)b] + b = a^3x + (a^2 + a + 1)b$		M1	Doesn't have to be correct yet
Then $a^3x + (a^2 + a + 1)b - 3ax - 3b + 2x \equiv 0$			
$(a^3 - 3a + 2)x + (a^2 + a - 2)b \equiv 0$		M1	Equating both coefft. of x and constant terms to zero
$(a - 1)(a^2 + a - 2)x + (a^2 + a - 2)b \equiv 0$		M1	Factorising
$(a^2 + a - 2)[(a - 1)x + b] \equiv 0$			
$(a + 2)(a - 1)[(a - 1)x + b] \equiv 0$			
We have, then, that $a = -2$ or 1 .		A1	Both a values correct
In either case, b takes any (arbitrary) value			
Solutions are thus $p_1(x) = -2x + b$ and $p_2(x) = x + b$		A1 A1	Must be arbitrary b .
Give one A1 if any one correct, but not both A1 s if extra answers appear.			7 marks

(ii)	Deg[RHS] = 4 while Deg[LHS] $\leq \max(n^2, 2n, n)$, so it follows that $n = 2$ and $p(x)$ is quadratic. Setting $p(x) = ax^2 + bx + c$, we have	E1	Supporting reasoning
		B1	Noted or implied anywhere
	$2p(p(x)) = 2a(ax^2 + bx + c)^2 + 2b(ax^2 + bx + c) + 2c$	M1	
	$= 2a\{a^2x^4 + 2abx^3 + 2acx^2 + b^2x^2 + 2bcx + c^2\} + 2b(ax^2 + bx + c) + 2c$		
	$3(p(x))^2 = 3[a^2x^4 + 2abx^3 + (2ac + b^2)x^2 + 2bcx + c^2]$	M1	
	& $-4p(x) = -4ax^2 - 4bx - 4c$		
	LHS = $(2a^3 + 3a^2)x^4 + (4a^2b + 6ab)x^3 + (2ab^2 + 4a^2c + 2ab + 3b^2 + 6ac - 4a)x^2$ $+ (4abc + 2b^2 + 6bc - 4b)x + (2ac^2 + 2bc + 2c + 3c^2 - 4c)$		
	RHS = x^4		
	Equating terms:	M1	
	$x^4) \quad 2a^3 + 3a^2 - 1 = 0 \Rightarrow (a+1)^2(2a-1) \Rightarrow a = -1 \text{ or } \frac{1}{2}$	A1	From correct terms
	$x^3) \quad 2ab(2a+3) = 0 \Rightarrow b = 0$	A1	From correct terms
	$x^2) \quad 2a(2ac + 3c - 2) = 0 \Rightarrow c = 2 \text{ when } a = -1$	A1	i.e. $p_1(x) = -x^2 + 2$
	OR $c = \frac{1}{2} \text{ when } a = \frac{1}{2}$	A1	i.e. $p_2(x) = \frac{1}{2}(x^2 + 1)$
	$x^1) \quad 2b(2ac + b + 3c - 2) = 0$ Checks		
	$x^0) \quad c(2ac + 3c - 2) = 0$ Checks	E1	Both checks must be visible
10 marks			

STEP II 2012 Q3

$t = \sqrt{x^2 + 1} + x \Rightarrow \frac{1}{t} = \sqrt{x^2 + 1} - x$ and $x = \frac{1}{2}\left(t - \frac{1}{t}\right)$	M1	
OR via $(t-x)^2 = x^2 + 1 \Rightarrow t^2 - 2tx = 1 \Rightarrow x = \frac{t^2 - 1}{2t}$ or $x = \frac{1}{2}t - \frac{1}{2}t^{-1}$	A1	
$dx = \left(\frac{1}{2} + \frac{1}{2}t^{-2}\right) dt$	B1	
Also $x : (0, \infty) \rightarrow t : (1, \infty)$ some stage	B1	Limits dealt with at
so that $\int_0^\infty f(\sqrt{x^2 + 1} + x) dx = \int_1^\infty f(t) \times \frac{1}{2} \left(1 + \frac{1}{t^2}\right) dt$	M1	Full substitution
$= \frac{1}{2} \int_1^\infty f(x) \left(1 + \frac{1}{x^2}\right) dx$	A1 Answer Given	6 marks

$$I_1 = \int_0^{\infty} \frac{1}{\left(\sqrt{x^2+1}+x\right)^2} dx \quad \text{i.e. } f(x) = \frac{1}{x^2} \quad \mathbf{M1}$$

NB Qn. says “Hence” so alternative methods not accepted unless they are using the previous substitution again

$$= \frac{1}{2} \int_1^{\infty} \left(1 + \frac{1}{x^2}\right) \cdot \frac{1}{x^2} dx = \frac{1}{2} \int_1^{\infty} \left(x^{-2} + x^{-4}\right) dx \quad \mathbf{A1}$$

$$= \frac{1}{2} \left[-\frac{1}{x} - \frac{1}{3x^3} \right]_1^{\infty} \quad \mathbf{A1} \quad \text{Integration correct}$$

$$= \frac{1}{2} \left(0 + 1 + \frac{1}{3}\right) = \frac{2}{3} \quad \mathbf{A1} \quad \mathbf{4 \text{ marks}}$$

$x = \tan \theta \Rightarrow dx = \sec^2 \theta d\theta$ **B1** Qn. says to use this
 substn. so alternative methods not accepted

$$\sqrt{1+x^2} = \sec \theta \quad \mathbf{B1} \quad \text{Used at any stage}$$

Limits: $\left(0, \frac{1}{2}\pi\right) \rightarrow (0, \infty)$ **B1**

$$I_2 = \int_0^{\frac{1}{2}\pi} \frac{1}{(1+\sin \theta)^3} d\theta = \int_0^{\frac{1}{2}\pi} \left(\frac{\sec \theta}{\sec \theta + \tan \theta} \right)^3 d\theta \quad \mathbf{M1}$$

$$= \int_0^{\frac{1}{2}\pi} \frac{\sec \theta}{(\sec \theta + \tan \theta)^3} \cdot \sec^2 \theta d\theta = \int_0^{\infty} \frac{\sqrt{x^2+1}}{\left(\sqrt{x^2+1}+x\right)^3} dx \quad \mathbf{A1}$$

$$f(t) = \frac{\frac{1}{2} \left(t + \frac{1}{t} \right)}{t^3} = \frac{t^2+1}{2t^4} \quad \mathbf{M1}$$

$$= \frac{1}{2} \int_1^{\infty} \left(\frac{t^2+1}{t^2} \right) \left(\frac{t^2+1}{2t^4} \right) dt \quad \mathbf{A1}$$

$$= \frac{1}{4} \int_1^{\infty} \left(t^{-2} + 2t^{-4} + t^{-6} \right) dt \quad \mathbf{A1}$$

$$= \frac{1}{4} \left[-\frac{1}{t} - \frac{2}{3t^3} - \frac{1}{5t^5} \right]_1^{\infty} \quad \mathbf{A1} \quad \text{Integration correct}$$

$$= \frac{1}{4} \left(0 + 1 + \frac{2}{3} + \frac{1}{5} \right) = \frac{7}{15} \quad \mathbf{A1} \quad \mathbf{10 \text{ marks}}$$

STEP II 2012 Q4

(i) $n, k > 1 \Rightarrow n^{k+1} > n^k$ and $k+1 > k$ so $(k+1) \times n^{k+1} > k \times n^k$ **E1**

$$\Rightarrow \frac{1}{(k+1)n^{k+1}} < \frac{1}{kn^k} \quad \text{E1} \quad (\text{since all terms} > 0)$$

2 marks

$$\ln\left(1 + \frac{1}{n}\right) = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \frac{1}{5n^5} - \dots \quad \text{valid as } 0 < \frac{1}{n} < 1 \quad \text{E1}$$

$$= \frac{1}{n} - \left(\frac{1}{2n^2} - \frac{1}{3n^3}\right) - \left(\frac{1}{4n^4} - \frac{1}{5n^5}\right) - \dots \quad \text{M1}$$

$$< \frac{1}{n} \quad \text{since each bracketed term is positive} \quad \text{A1}$$

by the previous result

E1 Justification that each bracket positive

$$\Rightarrow 1 + \frac{1}{n} < e^{\frac{1}{n}} \Rightarrow \left(1 + \frac{1}{n}\right)^n < e$$

B1 **5 marks**

ALT. Max 4/5 for non-“Hence” methods; e.g. using the exponential series

(ii) $\ln\left(\frac{2y+1}{2y-1}\right) = \ln\left(1 + \frac{1}{2y}\right) - \ln\left(1 - \frac{1}{2y}\right) \quad \text{M1} \quad \text{Log. work}$

$$= \left(\frac{1}{2y} - \frac{1}{2(2y)^2} + \frac{1}{3(2y)^3} - \frac{1}{4(2y)^4} + \frac{1}{5(2y)^5} - \dots\right) - \left(-\frac{1}{2y} - \frac{1}{2(2y)^2} - \frac{1}{3(2y)^3} - \frac{1}{4(2y)^4} - \frac{1}{5(2y)^5} - \dots\right)$$

M1 A1 A1 Use of log. series (given in question); 1st correct; 2nd correct

$$= 2\left(\frac{1}{2y} + \frac{1}{3(2y)^3} + \frac{1}{5(2y)^5} + \dots\right) \quad \text{A1} \quad \text{All correct}$$

$$> \frac{1}{y} \quad (\text{since following terms all positive}) \quad \text{A1}$$

Series valid for $0 < \frac{1}{2y} < 1$ i.e. $y > \frac{1}{2}$

E1 **7 marks**

Then $\ln\left(\frac{2y+1}{2y-1}\right)^y > 1 \quad \text{B1}$

Setting $y = n + \frac{1}{2}$ **M1** $\Rightarrow \ln\left(\frac{2n+2}{2n}\right)^{n+\frac{1}{2}} > 1 \Rightarrow \left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}} > e$ **A1 A1** **4 marks**

(iii) As $n \rightarrow \infty$, $\left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}} = \left(1 + \frac{1}{n}\right)^n \times \left(1 + \frac{1}{n}\right)^{\frac{1}{2}} \rightarrow \left(1 + \frac{1}{n}\right)^n \times 1 + \quad \text{E1}$

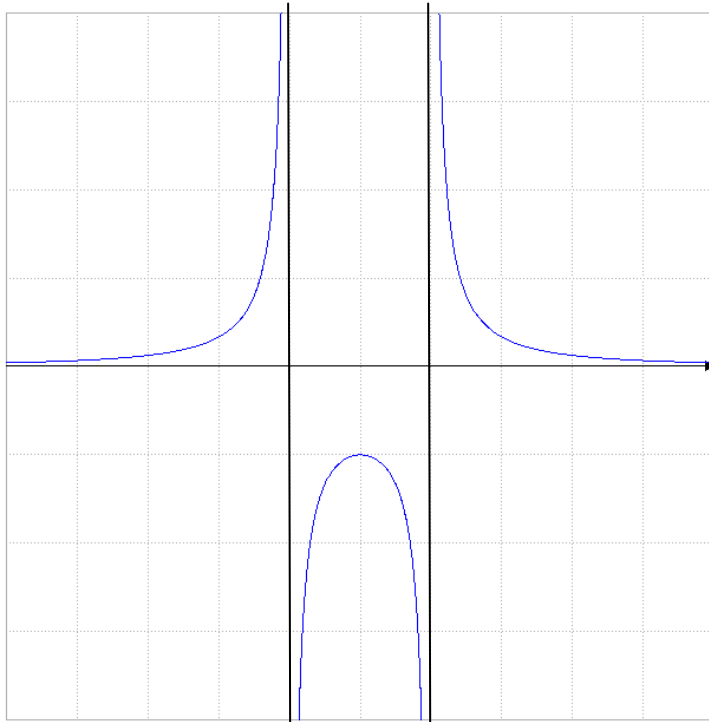
i.e. $\rightarrow \left(1 + \frac{1}{n}\right)^n$ from above and e is squeezed into the same limit from both above and below

E1 Informal explanation only required

2 marks

STEP II 2012 Q5

(i)



G1 Vertical asymptotes at $x = a - 1$ and $x = a + 1$

G1 Horizontal asymptote the x -axis

G1 Symmetry in $x = a$

G1 Three branches:
LH and RH branches $\approx 1/x^2$;
middle branch $\cap$ -shaped
(with max. at $x \approx a$)

Ignore the position of the y -axis

4 marks

(ii)
$$g'(x) = \frac{-2}{[(x-a)^2 - 1]^2 [(x-b)^2 - 1]^2} \{(x-b)[(x-a)^2 - 1] + (x-a)[(x-b)^2 - 1]\}$$

M1 Differentiated (may be done implicitly after “logging”)

Setting the numerator = 0 **M1**

$$(x-a)(x-b)[x-a+x-b] + [x-a+x-b] = 0$$

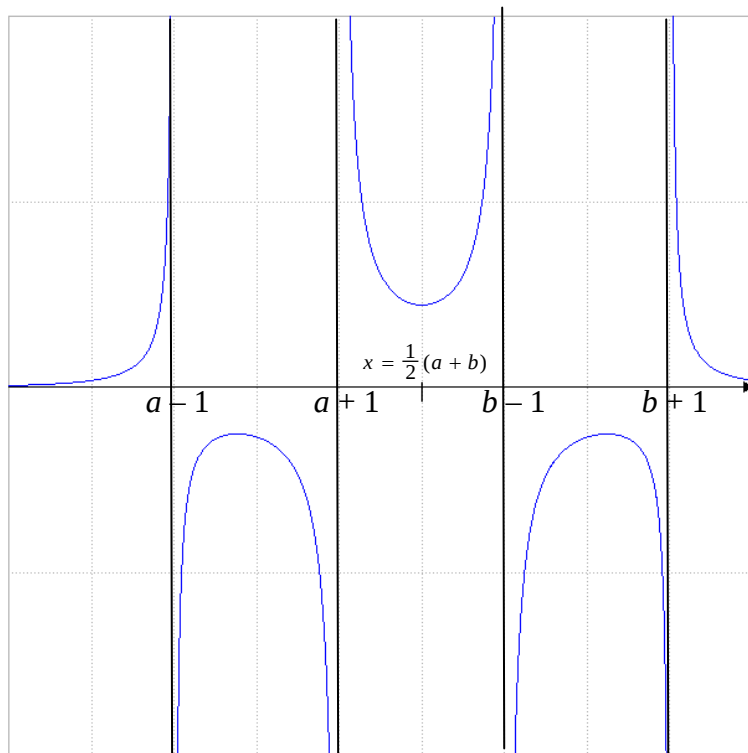
M1 Sensible factorisation attempt: $(2x-a-b)(x^2 - (a+b)x + (ab-1)) = 0$

$$x = \frac{1}{2}(a+b) \quad \text{A1} \quad \text{or} \quad x = \frac{a+b \pm \sqrt{(a+b)^2 - 4ab + 4}}{2}$$

A1 A1 Any sensible form: e.g. $x = \frac{1}{2} \{a+b \pm \sqrt{(b-a)^2 + 4}\}$

6 marks

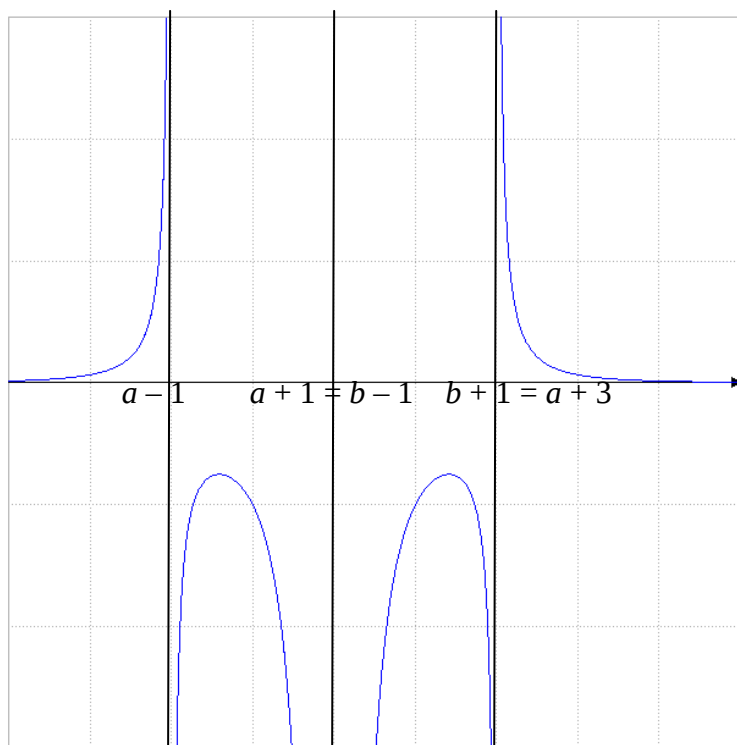
Case 1 $b > a + 2$ (i.e. $a + 1 < b - 1$)



- G1** Five branches
- G1** 4 vertical asymptotes (with correct coordinates)
- G1** LH & RH branches correctly placed
- G1** Middle three branches correctly placed
- G1** All correct

5 marks

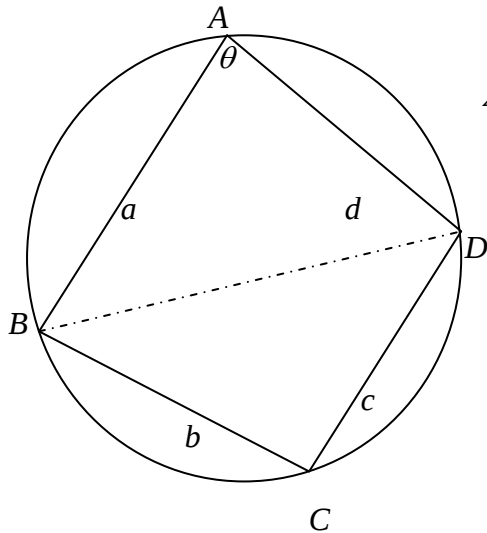
Case 2 $b = a + 2$ (i.e. $a + 1 = b - 1$)



- G1** Four branches
- G1** 3 vertical asymptotes (with correct coordinates)
- G1** LH & RH branches correctly placed
- G1** Middle two branches correctly placed
- G1** All correct

5 marks

STEP II 2012 Q6



$$\angle BCD = \pi - \theta \quad (\text{Opp. } \angle\text{s cyclic quad.})$$

B1 Noted or used
(possibly implicitly)

Cosine Rule in $\triangle BAD$:

$$BD^2 = a^2 + d^2 - 2ad \cos \theta$$

M1

A1

Cosine Rule in $\triangle BCD$:

$$BD^2 = b^2 + c^2 + 2bc \cos \theta$$

M1

A1

Equating for BD^2

M1

Identifying $\cos \theta$

M1

$$= \frac{a^2 - b^2 - c^2 + d^2}{2(ad + bc)}$$

A1

8 marks

$$Q = \frac{1}{2} ad \sin \theta + \frac{1}{2} bc \sin \theta$$

M1 Since $\sin(\pi - \theta) = \sin \theta$

$$\Rightarrow \sin \theta = \frac{2Q}{ad + bc} \text{ or } \frac{4Q}{2(ad + bc)}$$

A1

2 marks

$$\text{Use of } \sin^2 \theta + \cos^2 \theta = 1 \Rightarrow \frac{16Q^2}{4(ad + bc)^2} + \frac{(a^2 - b^2 - c^2 + d^2)^2}{4(ad + bc)^2} = 1 \quad \mathbf{M1}$$

$$\Rightarrow 16Q^2 = 4(ad + bc)^2 - (a^2 - b^2 - c^2 + d^2)^2$$

A1

Answer Given

2 marks

$$16Q^2 = (2ad + 2bc - a^2 + b^2 + c^2 - d^2)(2ad + 2bc + a^2 - b^2 - c^2 + d^2) \quad \mathbf{M1} \quad \text{Use of the difference of 2 squares factorisation}$$

$$= ([b + c]^2 - [a - d]^2)([a + d]^2 - [b - c]^2)$$

M1

Completing squares

$$= ([b + c] - [a - d])([b + c] + [a - d])([a + d] - [b - c])([a + d] + [b - c])$$

M1 Use of the difference of 2 squares factorisation in both brackets

$$= (b + c + d - a)(a + b + c - d)(a + c + d - b)(a + b + d - c)$$

A1

Splitting the 16 into four 2's (one per bracket) and using $2s = a + b + c + d$

M1

$$\Rightarrow Q^2 = \frac{(2s - 2a)}{2} \frac{(2s - 2b)}{2} \frac{(2s - 2c)}{2} \frac{(2s - 2d)}{2} = (s - a)(s - b)(s - c)(s - d)$$

A1

Answer Given

6 marks

For a triangle (guaranteed cyclic) let $d \rightarrow 0$ (**Or** $s - d \rightarrow s$ **Or** let $D = A$)

E1

$$\Delta = \sqrt{s(s - a)(s - b)(s - c)} \quad \text{with or without explanation}$$

B1

2 marks

STEP II 2012 Q7

Centroid, G , has p.v. $\mathbf{g} = \frac{1}{3}(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3)$

$$\overrightarrow{GX_1} = \mathbf{x}_1 - \mathbf{g} = \frac{1}{3}(2\mathbf{x}_1 - \mathbf{x}_2 - \mathbf{x}_3) \quad \text{M1 A1}$$

$$\text{and so } \overrightarrow{GY_1} = -\frac{1}{3}\lambda_1(2\mathbf{x}_1 - \mathbf{x}_2 - \mathbf{x}_3) \quad (\text{where } \lambda_1 > 0) \quad \text{B1}$$

$$\text{Then } \overrightarrow{OY_1} = \overrightarrow{OG} + \overrightarrow{GY_1} = \frac{1}{3}(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3) - \frac{1}{3}\lambda_1(2\mathbf{x}_1 - \mathbf{x}_2 - \mathbf{x}_3) \quad \text{M1}$$

$$= \frac{1}{3}([1 - 2\lambda_1]\mathbf{x}_1 + [1 + \lambda_1](\mathbf{x}_2 + \mathbf{x}_3)) \quad \text{A1} \quad \text{5 marks}$$

Circle centre O , radius 1 has equation $|\mathbf{x}|^2 = 1$ or $\mathbf{x} \cdot \mathbf{x} = 1$ noted at any stage **B1**

Since $\overrightarrow{OY_1} \cdot \overrightarrow{OY_1} = 1$, we have **M1**

(Note: using $\alpha = \mathbf{x}_2 \cdot \mathbf{x}_3$, $\beta = \mathbf{x}_3 \cdot \mathbf{x}_1$ and $\gamma = \mathbf{x}_1 \cdot \mathbf{x}_2$)

$$1 = \frac{1}{9} \{ (1 - 2\lambda_1)^2 + 2(1 + \lambda_1)^2 + 2(1 - 2\lambda_1)(1 + \lambda_1)\mathbf{x}_1 \cdot (\mathbf{x}_2 + \mathbf{x}_3) + 2(1 + \lambda_1)^2 \mathbf{x}_2 \cdot \mathbf{x}_3 \} \quad \text{A1}$$

$$\Rightarrow 9 = 1 - 4\lambda_1 + 4\lambda_1^2 + 2 + 4\lambda_1 + 2\lambda_1^2 + 2(1 - 2\lambda_1)(1 + \lambda_1)\mathbf{x}_1 \cdot (\mathbf{x}_2 + \mathbf{x}_3) + 2(1 + \lambda_1)^2 \mathbf{x}_2 \cdot \mathbf{x}_3 \quad \text{A1}$$

$$\Rightarrow 0 = -3(1 - \lambda_1)(1 + \lambda_1) + (1 - 2\lambda_1)(1 + \lambda_1)\mathbf{x}_1 \cdot (\mathbf{x}_2 + \mathbf{x}_3) + (1 + \lambda_1)^2 \mathbf{x}_2 \cdot \mathbf{x}_3 \quad \text{A1}$$

$$\text{As } \lambda_1 > 0, \quad 0 = -3(1 - \lambda_1) + (1 - 2\lambda_1)\mathbf{x}_1 \cdot (\mathbf{x}_2 + \mathbf{x}_3) + (1 + \lambda_1)\mathbf{x}_2 \cdot \mathbf{x}_3 \quad \text{M1}$$

$$\Rightarrow 0 = -3 + 3\lambda_1 + (\mathbf{x}_1 \cdot \mathbf{x}_2 + \mathbf{x}_2 \cdot \mathbf{x}_3 + \mathbf{x}_3 \cdot \mathbf{x}_1) + \lambda_1(\mathbf{x}_2 \cdot \mathbf{x}_3) - 2\lambda_1(\mathbf{x}_1 \cdot \mathbf{x}_2 + \mathbf{x}_1 \cdot \mathbf{x}_3) \quad \text{M1}$$

$$\Rightarrow \lambda_1 = \frac{3 - (\alpha + \beta + \gamma)}{3 + \alpha - 2\beta - 2\gamma} \quad \text{A1} \quad \text{Answer Given} \quad \text{8 marks}$$

$$\text{Similarly, } \lambda_2 = \frac{3 - (\alpha + \beta + \gamma)}{3 + \beta - 2\alpha - 2\gamma} \quad \text{and} \quad \lambda_3 = \frac{3 - (\alpha + \beta + \gamma)}{3 + \gamma - 2\alpha - 2\beta} \quad \text{B1 B1}$$

$$\frac{GX_i}{GY_i} = \frac{1}{\lambda_i} \quad \text{noted or used} \quad \text{B1}$$

$$\frac{GX_1}{GY_1} + \frac{GX_2}{GY_2} + \frac{GX_3}{GY_3} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \quad \text{M1}$$

$$= \frac{9 + (\alpha + \beta + \gamma) - 4(\alpha + \beta + \gamma)}{3 - (\alpha + \beta + \gamma)} \quad \text{M1}$$

$$= \frac{9 - 3(\alpha + \beta + \gamma)}{3 - (\alpha + \beta + \gamma)} = 3 \quad \text{M1 A1 Factorising leading to Given Answer} \quad \text{7 marks}$$

NB This result generalises to n points on a circle: $\sum_{i=1}^n \frac{GX_i}{GY_i} = n$.

STEP II 2012 Q8

$$\beta - \alpha > q (> 0) \Rightarrow \beta^2 - 2\alpha\beta + \alpha^2 > q^2 \quad \text{M1}$$

$$\Rightarrow \alpha^2 + \beta^2 - q^2 > 2\alpha\beta \Rightarrow \frac{\alpha^2 + \beta^2 - q^2}{\alpha\beta} > 2 \Rightarrow \frac{\alpha^2 + \beta^2 - q^2}{\alpha\beta} - 2 > 0$$

A1 *Answer Given* ($\alpha, \beta > 0$ given)

2 marks

$$u_{n+1} = \frac{u_n^2 - q^2}{u_{n-1}} \text{ etc. } \Rightarrow u_n^2 - u_{n+1}u_{n-1} = q^2 = u_{n+1}^2 - u_{n+2}u_n \quad \text{M1} \quad \text{Equating for } q^2$$

$$\Rightarrow u_n(u_n + u_{n+2}) = u_{n+1}(u_{n-1} + u_{n+1}) \quad \text{A1} \quad \text{Re-arranged \& factorised to get } \textbf{Given Answer} \quad \text{2 marks}$$

$$\text{Then } \frac{u_n + u_{n+2}}{u_{n+1}} = \frac{u_{n-1} + u_{n+1}}{u_n} \quad \text{B1}$$

$$\text{which } \Rightarrow \frac{u_{n-1} + u_{n+1}}{u_n} \text{ is constant (independent of } n) \quad \text{M2}$$

$$\text{Calling this constant } p \text{ gives } u_{n+1} - pu_n + u_{n-1} = 0, \text{ as required} \quad \text{A1} \quad \textbf{Answer Given}$$

$$u_2 = \frac{\beta^2 - q^2}{\alpha} \quad \text{B1} \quad \text{Anywhere}$$

$$p = \frac{u_0 + u_2}{u_1} = \frac{\alpha + \frac{\beta^2 - q^2}{\alpha}}{\beta} \quad \text{M1} \quad \text{Use of first terms (even if not proved this is } \textit{always} \text{ the constant } p)$$

$$= \frac{\alpha^2 + \beta^2 - q^2}{\alpha\beta} \quad \text{A1} \quad \text{7 marks}$$

ALTERNATIVE METHOD

$$u_2 = \gamma = \frac{\beta^2 - q^2}{\alpha} = p\beta - \alpha \Leftrightarrow p = \frac{\alpha^2 + \beta^2 - q^2}{\alpha\beta} \quad \text{M1 A1}$$

$$u_3 = \frac{\gamma^2 - q^2}{\beta} = p\gamma - \beta \Leftrightarrow p = \frac{\gamma^2 + \beta^2 - q^2}{\beta\gamma} \quad \text{M1 A1}$$

$$= \frac{\left(\frac{\beta^2 - q^2}{\alpha}\right)^2 + \beta^2 - q^2}{\beta\left(\frac{\beta^2 - q^2}{\alpha}\right)} \quad \text{M1} \quad = \frac{(\beta^2 - q^2)^2 + \alpha^2(\beta^2 - q^2)}{\alpha\beta(\beta^2 - q^2)} = \frac{\beta^2 - q^2 + \alpha^2}{\alpha\beta} \quad \text{A1} \quad \text{since}$$

$\beta^2 - q^2 \neq 0$ as u_2 non-zero (given). Since p is consistent for any chosen α, β , the proof follows inductively on any two consecutive terms of the sequence. **E1**

7 marks

If $\beta > \alpha + q$, $u_{n+1} - u_n = (p-1)u_n - u_{n-1} = \left(\frac{\beta^2 + \alpha^2 - q^2}{\alpha\beta} - 1 \right) u_n - u_{n-1}$

M1 Considering $u_{n+1} - u_n$; **M1** using p

$> (2-1)u_n - u_{n-1}$ by the initial result **M1**

$> u_n - u_{n-1}$

Hence, if $u_n - u_{n-1} > 0$ then so is $u_{n+1} - u_n$ **A1** Valid conclusion

Since $\beta > \alpha$, $u_2 - u_1 > 0$ and proof follows by induction **E1** **5 marks**

If $\beta = \alpha + q$ then $p = 2$ **B1**

and $u_{n+1} - u_n = u_n - u_{n-1}$ so that the sequence is an AP **B1**

Also, $u_0 = \alpha$, $u_1 = \alpha + q$, $u_2 = \alpha + 2q$, ... $\Rightarrow$ common difference is q

M1 A1 (which is still a strictly increasing sequence since $q > 0$ given) **4 marks**

STEP II 2012 Q9

Use of $x = ut \cos \alpha$ **M1** When $x = a$, $t = \frac{a}{u \cos \alpha}$ **A1**

$y = 2h - ut \sin \alpha - \frac{1}{2}gt^2$ **(*) B1**

Subst^g. in their t at $x = a$ into their y **M1** $y = 2h - a \tan \alpha - \frac{ga^2}{2u^2} \sec^2 \alpha$

A1 May still be $\cos^2 \alpha$ at this stage

Use of their $y > h$ **M1** $2h - a \tan \alpha - \frac{ga^2}{2u^2} \sec^2 \alpha > h \Rightarrow h - a \tan \alpha > \frac{ga^2}{2u^2} \sec^2 \alpha$

$\Rightarrow \frac{1}{u^2} < \frac{2(h - a \tan \alpha)}{ga^2 \sec^2 \alpha}$ **A1** **Answer Given** **7 marks**

Setting $y = 0$ in **(*)** and writing it as a quadratic in t **M1**

$t = \frac{-2u \sin \alpha + \sqrt{4u^2 \sin^2 \alpha + 16gh}}{2g}$ **A1** No need to mention that the negative root is inappropriate; allow $\pm$ for now

Setting $x = u \cos \alpha \times$ their t **M1**

Setting their $x < b$ **M1** $u \cos \alpha \left(\frac{\sqrt{u^2 \sin^2 \alpha + 4gh} - u \sin \alpha}{g} \right) < b$

$\Rightarrow \sqrt{u^2 \sin^2 \alpha + 4gh} < \frac{bg}{u \cos \alpha} + u \sin \alpha$ **A1** **Answer Given** **5 marks**

M1 Dividing throughout by $\cos \alpha$: $\sqrt{u^2 \tan^2 \alpha + 4gh \sec^2 \alpha} < \frac{bg \sec^2 \alpha}{u} + u \tan \alpha$

M1 Squaring (both sides positive):

$$u^2 \tan^2 \alpha + 4gh \sec^2 \alpha < \frac{b^2 g^2 \sec^4 \alpha}{u^2} + 2bg \sec^2 \alpha \tan \alpha + u^2 \tan^2 \alpha$$

M1 Cancelling $u^2 \tan^2 \alpha$ both sides & dividing by $g \sec^2 \alpha$: $4h < \frac{b^2 g \sec^2 \alpha}{u^2} + 2b \tan \alpha$

M1 Using first result, $\frac{1}{u^2} < \frac{2(h - a \tan \alpha)}{a^2 g \sec^2 \alpha}$, in here: $4h < 2b \tan \alpha + b^2 g \sec^2 \alpha \times \frac{2(h - a \tan \alpha)}{ga^2 \sec^2 \alpha}$

M1 Cancelling and multiplying by a^2 : $4a^2 h < 2a^2 b \tan \alpha + 2b^2 (h - a \tan \alpha)$

M1 Re-arranging for $\tan \alpha$: $2ab(b - a) \tan \alpha < 2h(b^2 - 2a^2)$

A1 Answer $\tan \alpha < \frac{h(b^2 - 2a^2)}{ab(b - a)}$ legitimately obtained **Answer Given**

E1 Explanation that $b > a$ (other side of net) (else direction of inequality would reverse)

ALTERNATIVE

M1 Squaring $\sqrt{u^2 \sin^2 \alpha + 4gh} < \frac{bg}{u \cos \alpha} + u \sin \alpha$:

$$u^2 \sin^2 \alpha + 4gh < \frac{b^2 g^2 \sec^2 \alpha}{u^2} + 2bg \tan \alpha + u^2 \sin^2 \alpha$$

M1 Cancelling $u^2 \sin^2 \alpha$ both sides & dividing by g : $4h < \frac{b^2 g \sec^2 \alpha}{u^2} + 2b \tan \alpha$

M2 Re-arranging for $\frac{1}{u^2}$: $\frac{2(2h - b \tan \alpha)}{b^2 g \sec^2 \alpha} < \frac{1}{u^2}$

M1 Using first result, $\frac{1}{u^2} < \frac{2(h - a \tan \alpha)}{a^2 g \sec^2 \alpha}$, in here: $\frac{2(2h - b \tan \alpha)}{b^2 g \sec^2 \alpha} < \frac{2(h - a \tan \alpha)}{a^2 g \sec^2 \alpha}$

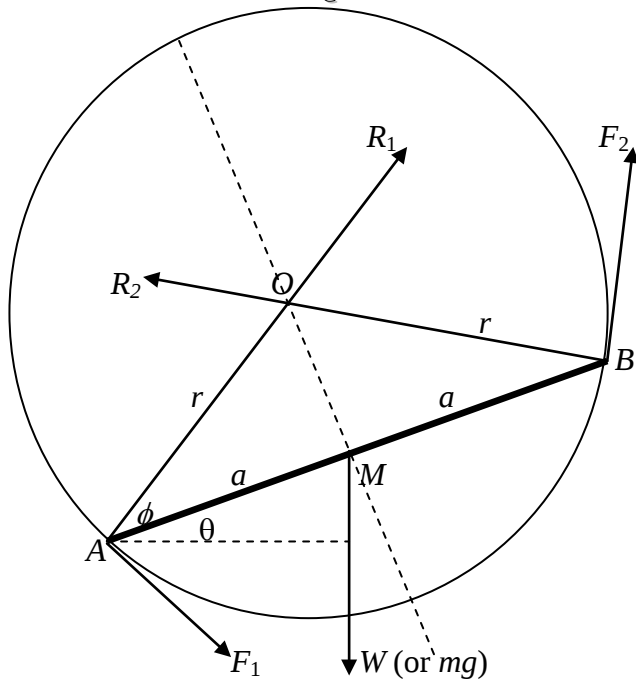
M1 Re-arranging for $\tan \alpha$: $ab(b - a) \tan \alpha < h(b^2 - 2a^2)$

A1 Answer $\tan \alpha < \frac{h(b^2 - 2a^2)}{ab(b - a)}$ legitimately obtained **Answer Given**

E1 Explanation that $b > a$ (other side of net) (else direction of inequality would reverse)

8 marks

STEP II 2012 Q10

Moments about M :

$$R_1 a \sin \phi = R_2 a \sin \phi + F_1 a \cos \phi + F_2 a \cos \phi$$

M1 Four terms

A1 Correct magnitudes

A1 Correct signs

Friction Law: $F_1 = \mu R_1$ and $F_2 = \mu R_2$ **B1**

Dividing by $\cos \phi$ and re-arranging **M1**

$$R_1 \tan \phi = R_2 \tan \phi + \mu R_1 + \mu R_2$$

$$\Rightarrow (R_1 - R_2) \tan \phi = \mu (R_1 + R_2)$$

A1

Answer Given

6 marks

Moments about O : $\mu (R_1 - R_2) r = W r \sin \phi \sin \theta$

M1 A1 A1

Method; LHS; RHS

Resolving // AB: $(R_1 - R_2) \cos \phi + \mu (R_1 + R_2) \sin \phi = W \sin \theta$

M1 A1 A1

Method; LHS; RHS

(Give one **A1** here if all correct apart from a – sign)

Resolving $\perp^r AB$: $(R_1 + R_2) \sin \phi - \mu (R_1 - R_2) \cos \phi = W \cos \theta$

M1 A1 A1

Method; LHS; RHS

(Give one **A1** here if all correct apart from a – sign)

NB – Only two of these are required (give for their two best efforts)

6 marks

M1 Dividing these last two eqns:

$$\tan \theta = \frac{(R_1 - R_2) \cos \phi + \mu(R_1 + R_2) \sin \phi}{(R_1 + R_2) \sin \phi - \mu(R_1 - R_2) \cos \phi}$$

M1 Use of first result, $\mu(R_1 + R_2) = (R_1 - R_2) \tan \phi$:

$$\tan \theta = \frac{(R_1 - R_2) \cos \phi + (R_1 - R_2) \tan \phi \sin \phi}{(R_1 - R_2) \frac{\tan \phi}{\mu} \sin \phi - \mu(R_1 - R_2) \cos \phi}$$

A1 $\tan \theta = \frac{\cos \phi + \tan \phi \sin \phi}{\frac{\tan \phi}{\mu} \sin \phi - \mu \cos \phi}$ No need to note that $R_1 \neq R_2$

M1 Multiplying throughout by $\mu \cos \phi$: $\tan \theta = \frac{\mu(\cos^2 \phi + \sin^2 \phi)}{\sin^2 \phi - \mu^2 \cos^2 \phi} = \frac{\mu}{1 - \cos^2 \phi - \mu^2 \cos^2 \phi}$

M1 Use of $\cos \phi = \frac{a}{r}$

A1 $\tan \theta = \frac{\mu}{1 - \frac{a^2}{r^2} - \mu^2 \left(\frac{a^2}{r^2} \right)} = \frac{\mu r^2}{r^2 - a^2 (1 + \mu^2)}$ **Answer Given**

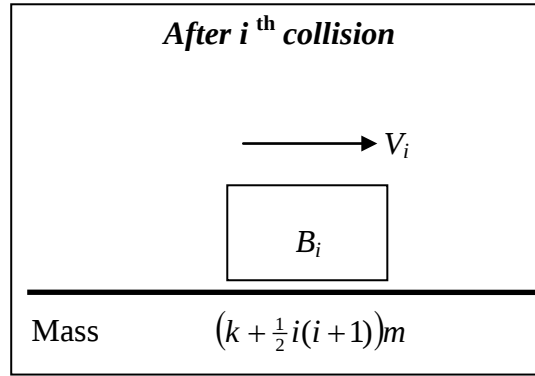
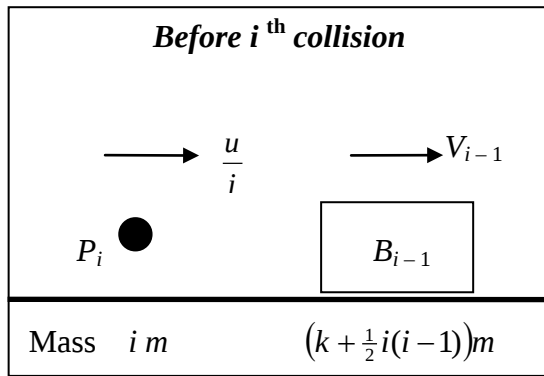
6 marks

M1 $\tan \lambda = \mu = \left(\frac{R_1 - R_2}{R_1 + R_2} \right) \tan \phi$ from the first result

 $\tan \phi < \lambda \Rightarrow \phi < \lambda$ **A1 Answer Given**

2 marks

STEP II 2012 Q11



Mass of block before/after

CLM $\rightarrow m u + M V_{i-1} = (M + im) V_i$

B1 B1 Allow unsimplified for now

M1 A1 This done at any stage (NB $V_0 = 0$)

M1 A1 This done at a general stage

$$V_1 = \frac{u}{k+1}, V_2 = \frac{2u}{k+1+2}, V_3 = \frac{3u}{k+1+2+3}, \dots$$

A1 ≥ 3 terms (or alt. method for generalising)

$$V_n = \frac{nu}{k + \frac{1}{2}n(n+1)} = \frac{2nu}{2k + n(n+1)}$$

A1 General term correct **Answer Given 8 marks**

ALTERNATIVE

CLM $\rightarrow$ for *all* particles $mu + 2m\left(\frac{u}{2}\right) + 3m\left(\frac{u}{3}\right) + \dots + nm\left(\frac{u}{n}\right) = (k + \frac{1}{2}n(n+1))mV$ **M2 A2 B1**

A1 Method; LHS; Final total mass; RHS **M1** Re-arranging for V **A1** for $V_n = \frac{2nu}{2k + n(n+1)}$

Last collision occurs when $V_n \geq \frac{u}{n+1}$ **M1**

i.e. $\frac{2nu}{N(N+1) + n(n+1)} \geq \frac{u}{n+1}$ **M1** Use of given k in first result

$\Rightarrow 2n(n+1) \geq N(N+1) + n(n+1) \Rightarrow n(n+1) \geq N(N+1)$ **M1** Re-arrangement

$\Rightarrow$ there are N collisions **A1** **4 marks**

Total KE of all the P_i 's is $\sum_{i=1}^N \frac{1}{2}(im)\left(\frac{u}{i}\right)^2 = \frac{1}{2}mu^2 \sum_{i=1}^N \frac{1}{i}$ **M1 A1**

Final KE of the block is $\frac{1}{2}N(N+1)mV_N^2$ **M1 A1** Correct final mass

$= \frac{1}{2}N(N+1)m\left(\frac{u}{N+1}\right)^2 = \frac{1}{2}mu^2\left(\frac{N}{N+1}\right)$ **A1** Correct final speed

Loss in KE is the difference: $\frac{1}{2}mu^2 \sum_{i=1}^N \frac{1}{i} - \frac{1}{2}mu^2\left(\frac{N}{N+1}\right)$ **M1**

Use of $\frac{N}{N+1} = 1 - \frac{1}{N+1}$ **M1**

Loss in KE $= \frac{1}{2}mu^2\left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{N} - 1 + \frac{1}{N+1}\right) = \frac{1}{2}mu^2 \sum_{i=2}^{N+1} \left(\frac{1}{i}\right)$ **A1** **Answer Given**

8 marks

STEP II 2012 Q12

$$P(\text{light on}) = p \times \frac{3}{4} \times \frac{1}{2} + (1-p) \times \frac{1}{4} \times \frac{1}{2}$$

$$= \frac{1}{8}(1+2p)$$

M1 A1

M1 Complementary probs.

A1

Then $P(\text{Hall} | \text{on}) = \frac{\frac{1}{8}(1-p)}{\frac{1}{8}(1+2p)}$ **M1** Conditional prob. **B1** Numr. correct **A1** *ft* Denomr.

$$= \frac{(1-p)}{(1+2p)} \quad \mathbf{A1} \text{ Answer Given}$$

8 marks

Recognition of $B(7, p_1)$ for some prob. p_1 **M1**

Modal value 3 $\Rightarrow \binom{7}{2}(p_1)^2(1-p_1)^5 < \binom{7}{3}(p_1)^3(1-p_1)^4$ and $\binom{7}{4}(p_1)^4(1-p_1)^3 < \binom{7}{3}(p_1)^3(1-p_1)^4$

B1 B1 B1 one for each correct binomial term (unsimplified)

M1 M1 for each correct inequality clearly stated (with some attempt to do something with them)

M1 for using $p_1 = \frac{(1-p)}{(1+2p)}$

M1 for using numerical binomial coeffs. and correct powers of (their) p_1 and $(1-p_1)$:

$$21\left(\frac{1-p}{1+2p}\right)^2\left(\frac{3p}{1+2p}\right)^5 < 35\left(\frac{1-p}{1+2p}\right)^3\left(\frac{3p}{1+2p}\right)^4 \Rightarrow 3(3p) < 5(1-p) \Rightarrow p < \frac{5}{14} \quad \mathbf{M1} \text{ Cancelling } \mathbf{A1}$$

correct RH half of the inequality **Answer Given**

$$35\left(\frac{1-p}{1+2p}\right)^4\left(\frac{3p}{1+2p}\right)^3 < 35\left(\frac{1-p}{1+2p}\right)^3\left(\frac{3p}{1+2p}\right)^4 \Rightarrow (1-p) < (3p) \Rightarrow p > \frac{1}{4} \quad \mathbf{M1} \text{ Cancelling } \mathbf{A1}$$

correct LH half of the inequality **Answer Given**

12 marks

STEP II 2012 Q13

$P(\text{no supermarkets}) = e^{-k\pi y^2}$ **M1** A single Poisson term **A1** Correct

2 marks

$P(Y < y) = 1 - e^{-k\pi y^2}$ **M1 A1**

Differentiating w.r.t. y **M1** $\Rightarrow f(y) = 2k\pi y e^{-k\pi y^2}$ **A1** *Answer Given*

4 marks

$E(Y) = \int_0^{\infty} 2k\pi y^2 e^{-k\pi y^2} dy$ **M1** *Integration by Parts attempted* **M1**

Writing $2k\pi y^2 e^{-k\pi y^2}$ as $y \left(2k\pi y e^{-k\pi y^2} \right)$ **M1**

$$\left[y \left(e^{-k\pi y^2} \right) \right]_0^{\infty} + \int_0^{\infty} e^{-k\pi y^2} dy \quad \mathbf{A1} \quad = 0 + \int_0^{\infty} e^{-k\pi y^2} dy$$

M1 Use of the substitution $x = y\sqrt{2k\pi}$ $= \frac{1}{\sqrt{2k\pi}} \int_0^{\infty} e^{-\frac{1}{2}x^2} dx$ **A1**

$$= \frac{1}{\sqrt{2k\pi}} \sqrt{\frac{\pi}{2}} = \frac{1}{2\sqrt{k}} \quad \mathbf{A1} \text{ (by the given result)} \quad \mathbf{7 marks}$$

$E(Y^2) = \int_0^{\infty} 2k\pi y^3 e^{-k\pi y^2} dy$ **M1** *Integration by Parts attempted* **M1**

Writing $2k\pi y^3 e^{-k\pi y^2}$ as $y^2 \left(2k\pi y e^{-k\pi y^2} \right)$ **M1**

$$\left[y^2 \left(e^{-k\pi y^2} \right) \right]_0^{\infty} + \int_0^{\infty} 2y e^{-k\pi y^2} dy \quad \mathbf{A1}$$

$$= 0 + \frac{1}{k\pi} \int_0^{\infty} 2k\pi y e^{-k\pi y^2} = \frac{-1}{k\pi} \left[e^{-k\pi y^2} \right]_0^{\infty} \quad \mathbf{M1} \text{ Use of a previous result (or a substitution)}$$

$$\Rightarrow E(Y^2) = \frac{1}{k\pi} \quad \mathbf{A1}$$

$$\Rightarrow \text{Var}(Y) = \frac{1}{k\pi} - \frac{1}{4k} = \frac{4 - \pi}{4k\pi} \quad \mathbf{A1} \text{ Answer Given}$$

7 marks