Mark Scheme (Results)

Summer 2017

Pearson Edexcel International A Level
in Further Pure Mathematics F3
(WFM03/01)

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General Marking Guidance

- All candidates must receive the same treatment. Examiners must mark the first candidate in exactly the same way as they mark the last.
- Mark schemes should be applied positively. Candidates must be rewarded for what they have shown they can do rather than penalised for omissions.
- Examiners should mark according to the mark scheme not according to their perception of where the grade boundaries may lie.
- There is no ceiling on achievement. All marks on the mark scheme should be used appropriately.
- All the marks on the mark scheme are designed to be awarded. Examiners should always award full marks if deserved, i.e. if the answer matches the mark scheme. Examiners should also be prepared to award zero marks if the candidate's response is not worthy of credit according to the mark scheme.
- Where some judgement is required, mark schemes will provide the principles by which marks will be awarded and exemplification may be limited.
- When examiners are in doubt regarding the application of the mark scheme to a candidate's response, the team leader must be consulted.
- Crossed out work should be marked UNLESS the candidate has replaced it with an alternative response.

EDEXCEL GCE MATHEMATICS

General Instructions for Marking

1. The total number of marks for the paper is 75.

2. The Edexcel Mathematics mark schemes use the following types of marks:

- **M** marks: method marks are awarded for 'knowing a method and attempting to apply it', unless otherwise indicated.
- **A** marks: Accuracy marks can only be awarded if the relevant method (M) marks have been earned.
- **B** marks are unconditional accuracy marks (independent of M marks)
- Marks should not be subdivided.

3. Abbreviations

These are some of the traditional marking abbreviations that will appear in the mark schemes.

- bod – benefit of doubt
- ft – follow through
- the symbol $\surd$ will be used for correct ft
- cao – correct answer only
- cso - correct solution only. There must be no errors in this part of the question to obtain this mark
- isw – ignore subsequent working
- awrt – answers which round to
- SC: special case
- oe – or equivalent (and appropriate)
- dep – dependent
- indep – independent
- dp decimal places
- sf significant figures
- $\ast$ The answer is printed on the paper
- ⎣ The second mark is dependent on gaining the first mark

4. All A marks are 'correct answer only' (cao.), unless shown, for example, as A1 ft to indicate that previous wrong working is to be followed through. After a misread however, the subsequent A marks affected are treated as A ft, but manifestly absurd answers should never be awarded A marks.

5. For misreading which does not alter the character of a question or materially simplify it, deduct two from any A or B marks gained, in that part of the question affected.

6. If a candidate makes more than one attempt at any question:
   - If all but one attempt is crossed out, mark the attempt which is NOT crossed out.
   - If either all attempts are crossed out or none are crossed out, mark all the attempts and score the highest single attempt.

7. Ignore wrong working or incorrect statements following a correct answer.

8. Be careful when scoring a response that is either all correct or all incorrect. It is very easy to click down the ‘0’ column when it was meant to be ‘1’ and all correct.

General Principles for Further Pure Mathematics Marking
(But note that specific mark schemes may sometimes override these general principles).

<u>Method mark for solving 3 term quadratic:</u>

1. Factorisation

$(x^2 + bx + c) = (x + p)(x + q)$, where $|pq| = |c|$,leading to x = …

$(ax^2 + bx + c) = (mx + p)(nx + q)$, where $|pq| = |c|$ and $|mn| = |a|$,leading to x = …

2. Formula

Attempt to use the correct formula (with values for a, b and c).

3. Completing the square

Solving $x^2 + bx + c = 0$: $\left(x \pm \frac{b}{2}\right)^2 \pm q \pm c = 0,\ \ q \neq 0$,leading to x = …

<u>Method marks for differentiation and integration:</u>

1. Differentiation

Power of at least one term decreased by 1. ($x^n \to x^{n-1}$)

2. Integration

Power of at least one term increased by 1. ($x^n \to x^{n+1}$)

Use of a formula

Where a method involves using a formula that has been learnt, the advice given in recent examiners' reports is that the formula should be quoted first.

Normal marking procedure is as follows:

Method mark for quoting a correct formula and attempting to use it, even if there are small errors in the substitution of values.

Where the formula is not quoted, the method mark can be gained by implication from correct working with values, but may be lost if there is any mistake in the working.

Exact answers

Examiners' reports have emphasised that where, for example, an exact answer is asked for, or working with surds is clearly required, marks will normally be lost if the candidate resorts to using rounded decimals.

Answers without working

The rubric says that these may not gain full credit. Individual mark schemes will give details of what happens in particular cases. General policy is that if it could be done "in your head", detailed working would not be required. Most candidates do show working, but there are occasional awkward cases and if the mark scheme does not cover this, please contact your team leader for advice.

Question Number	Scheme	Notes	Marks
1	$18\cosh x + 14\sinh x = 11 + e^x$		
	$18\left(\frac{e^x + e^{-x}}{2}\right) + 14\left(\frac{e^x - e^{-x}}{2}\right) = 11 + e^x$	Uses the correct exponential forms	M1
	$9e^{2x} + 9 + 7e^{2x} - 7 = 11e^x + e^{2x}$		
	$15e^{2x} - 11e^x + 2 \ (=0)$ or $15e^x - 11 + 2e^{-x} \ (=0)$	M1: Collects terms to obtain a 3 term equation A1:Correct equation in either of the forms shown	M1A1
	$(5e^x - 2)(3e^x - 1) = 0 \Rightarrow e^x = ...$ or $\left(5e^{\frac{x}{2}} - 2e^{\frac{-x}{2}}\right)\left(3e^{\frac{x}{2}} - e^{\frac{-x}{2}}\right)$ or $(5e^x - 2)(3 - e^{-x})$	Attempt to solve their 3TQ Depends on the second M mark	dM1
	$x = \ln\frac{2}{5}, \ \ln\frac{1}{3}$	Both; $\ln\frac{2}{5}$ or $\ln 0.4$; $\ln\frac{1}{3}$ or ln0.3rec -ln3 scores A0	A1
			(5)
			Total 5

Question Number	Scheme	Notes	Marks
2	$\mathbf{A}=\begin{pmatrix}-1 & 3 & a\\ 2 & 0 & 1\\ 1 & -2 & 1\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}2 & 0 & 4\\ 3 & -2 & 3\\ 1 & 2 & b\end{pmatrix}$		
(a)	$\mathbf{A}^{\mathrm{T}}=\begin{pmatrix}-1 & 2 & 1\\ 3 & 0 & -2\\ a & 1 & 1\end{pmatrix}$		B1
			(1)
(b)	$\mathbf{AB}=\begin{pmatrix}a+7 & 2a-6 & ab+5\\ 5 & 2 & b+8\\ -3 & 6 & b-2\end{pmatrix}$	M1: Correct attempt at **AB** Min 5 entries correct A1: Correct matrix	M1A1
			(2)
(c)	$(\mathbf{AB})^{\mathrm{T}}=\begin{pmatrix}a+7 & 5 & -3\\ 2a-6 & 2 & 6\\ ab+5 & b+8 & b-2\end{pmatrix}$	Transposed matrix must be seen	B1
	$\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}=\begin{pmatrix}2 & 3 & 1\\ 0 & -2 & 2\\ 4 & 3 & b\end{pmatrix}\begin{pmatrix}-1 & 2 & 1\\ 3 & 0 & -2\\ a & 1 & 1\end{pmatrix}$	Attempt $\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$ Must be in this order	
	$=\begin{pmatrix}a+7 & 5 & -3\\ 2a-6 & 2 & 6\\ ab+5 & b+8 & b-2\end{pmatrix}$	Must see matrices being multiplied (as line above) Min 5 entries correct, follow through their errors in transposing **A** and **B**	M1
	$\therefore (\mathbf{AB})^{\mathrm{T}}=\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$	Clearly shows $(\mathbf{AB})^{\mathrm{T}}=\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$ with conclusion and no errors Eg state $(\mathbf{AB})^{\mathrm{T}}=\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$ or connect through the working by = signs Or QED, hence shown, #, (list not exhaustive)	A1cso
			(3)
			Total 6

Question Number	Scheme	Notes	Marks
3	$y = x - \operatorname{artanh}\left(\dfrac{2x}{1+x^2}\right)$		
(a)	$\dfrac{\mathrm{d}\left\{\operatorname{artanh}\left(\dfrac{2x}{1+x^2}\right)\right\}}{\mathrm{d}x} = \dfrac{1}{1-\left(\dfrac{2x}{1+x^2}\right)^2} \times \left(\dfrac{2(1+x^2)-4x^2}{(1+x^2)^2}\right)$ M1: Correct attempt to differentiate artanh using the chain and quotient (or product) rule, obtaining an expression in terms of x A1: Correct expression in any form ie no simplification required Quotient rule must have denominator $(1+x^2)^2$ and numerator to be $\lambda(1+x^2)-\mu x^2,\ \lambda,\mu>0$ Product rule must reach $\dfrac{\lambda}{(1+x^2)} - \dfrac{\mu x^2}{(1+x^2)^2}\quad \lambda,\mu>0$		M1A1
	$1-\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1-\left(1+\dfrac{2}{x^2-1}\right) = \dfrac{2}{1-x^2}$	dM1: Attempt $1-\dfrac{\mathrm{d}y}{\mathrm{d}x}$ A1: cao and cso	dM1 A1cso
			(4)
	(a) Way 2		
	$y = x - \operatorname{artanh}\left(\frac{2x}{1+x^2}\right) = x - \frac{1}{2}\ln\left(\dfrac{1+\frac{2x}{1+x^2}}{1-\frac{2x}{1+x^2}}\right)$		
	$= x - \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)^2 = x - \ln(x+1) + \ln(x-1)$		
	$\dfrac{\mathrm{d}\{-\ln(x+1)+\ln(x-1)\}}{\mathrm{d}x} = \dfrac{-1}{x+1} + \dfrac{1}{x-1}$	M1: Attempt to differentiate A1: Correct derivative	M1A1
	$1-\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1-\left(1+\dfrac{2}{x^2-1}\right) = \dfrac{2}{1-x^2}$	dM1: Attempt $1-\frac{\mathrm{d}y}{\mathrm{d}x}$ A1: cao and cso	dM1 A1cso
	(a) Way 3		
	$y = x - \ln\left(\dfrac{x+1}{x-1}\right)$	Obtained as Way 2	
	$\dfrac{\mathrm{d}\left(\ln\left(\dfrac{x+1}{x-1}\right)\right)}{\mathrm{d}x} = \left(\dfrac{x-1}{x+1}\right) \times \dfrac{(x-1)-(x+1)}{(x-1)^2}$	M1: Attempt to differentiate A1: Correct derivative	M1A1
	$1-\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1-\left(1+\dfrac{2}{x^2-1}\right) = \dfrac{2}{1-x^2}$	dM1: Attempt $1-\frac{\mathrm{d}y}{\mathrm{d}x}$ A1: cao and cso	dM1 A1 cso

	(a) Way 4		
	$u=\text{artanh}\left(\dfrac{2x}{1+x^2}\right)\Rightarrow \tanh u=\dfrac{2x}{1+x^2}$		
	$\left(1-\tanh^2 u\right)\dfrac{\mathrm{d}u}{\mathrm{d}x}=\dfrac{2\left(1-x^2\right)}{\left(1+x^2\right)^2}$		
	$\dfrac{\mathrm{d}u}{\mathrm{d}x}=\dfrac{2\left(1-x^2\right)}{\left(1+x^2\right)^2}\times\dfrac{1}{1-\dfrac{4x^2}{\left(1+x^2\right)^2}}$	M1: Attempt to differentiate to obtain d(artanh(...))/dx as a function of x A1: Correct (unsimplified) derivative	
	Reduces to $\dfrac{2}{1-x^2}$		
	Then as main scheme	dM1A1cso	
(b)	$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=2\left(1-x^2\right)^{-2}\times -2x\left(=\dfrac{-4x}{\left(1-x^2\right)^2}\right)$	M1: Attempt second derivative (quotient/product rule as in (a))	M1A1ft
		A1ft: Follow through their k or leave as k	
	$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+x\left(1-\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2=\dfrac{-4x}{\left(1-x^2\right)^2}+x\left(\dfrac{2}{1-x^2}\right)^2=0$	M1: Attempt $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+x\left(1-\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2$ with their value for k from (a)	M1A1cso
		A1: cso	
			(4)
			Total 8

Question Number	Scheme	Notes	Marks
4	$\mathbf{M}=\begin{pmatrix}1&1&3\\1&5&1\\3&1&1\end{pmatrix}$		
(a)	$\begin{vmatrix}1-\lambda&1&3\\1&5-\lambda&1\\3&1&1-\lambda\end{vmatrix}(=0)$ $\Rightarrow(1-\lambda)\{(5-\lambda)(1-\lambda)-1\}-(1-\lambda-3)+3(1-3(5-\lambda))\ \ (=0)$ M1: Attempt characteristic equation (at least 2 'elements' correct) $\left[\text{"Elements" are } (1-\lambda)\{(5-\lambda)(1-\lambda)-1\},\ -(1-\lambda-3),\ +3(1-3(5-\lambda))\right]$		M1
	$\lambda=6\Rightarrow-5\times4+8+12=0$ or $\lambda^3-7\lambda^2+36=(\lambda-6)(\lambda^2-\lambda-6)\Rightarrow\lambda=6$	Verifies $\lambda=6$ is an eigenvalue or factorises cubic to $(\lambda-6)\times$quadratic and extracts $\lambda=6$	A1
	$(\lambda^2-\lambda-6)=0\Rightarrow\lambda=3,-2$	M1: Solves their 3 term quadratic or cubic $(\lambda-6)(\lambda^2+k\lambda\pm6)$ seen A1: Two other correct eigenvalues	M1A1
			(4)
ALT	Sub $\lambda=6$ into $\lvert\mathbf{M}-\lambda\mathbf{I}\rvert$ and shows this = 0 with no further work scores M1A1M0A0 For a "factor theorem" solution (ie sub further values for λ), one further correct value scores M1A1M1A0. Both further correct values scores 4/4		
	Solutions without working: (calculator?) M1A1 as above; M1A1 correct values or M0A0 one or both incorrect		
(b)	$\begin{pmatrix}1&1&3\\1&5&1\\3&1&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=6\begin{pmatrix}x\\y\\z\end{pmatrix}$ or $\begin{pmatrix}-5&1&3\\1&-1&1\\3&1&-5\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ M1: Either statement is sufficient or equivalent in equation form		M1
	$x+y+3z=6x, x+5y+z=6y, 3x+y+z=6z$ $\Rightarrow x=\ldots$ or $y=\ldots$ or $z=\ldots$	Solves two equations to obtain one variable in terms of another	dM1
	$\begin{pmatrix}1\\2\\1\end{pmatrix}$ or $x=k,\ y=2k,\ z=k\ \ k\neq0$	Any multiple	A1
	$\pm\begin{pmatrix}\frac{1}{\sqrt{6}}\\\frac{2}{\sqrt{6}}\\\frac{1}{\sqrt{6}}\end{pmatrix}$ oe	Correct normalised vector Can be positive or negative Can be written in the **i**,**j**,**k** form	A1 **(4)**
			Total 8

Question Number	Scheme	Notes	Marks
5	$I_n = \int \text{cosec}^n x \,\mathrm{d}x = \int \text{cosec}^{n-2} x \,\text{cosec}^2 x \,\mathrm{d}x$		
(a)	$= -\cot x \,\text{cosec}^{n-2} x - \int (n-2)\text{cosec}^{n-2} x \cot^2 x \,\mathrm{d}x$	M1: Parts in the correct direction A1: Correct unsimplified expression	M1A1
	$= -\cot x \,\text{cosec}^{n-2} x - \int (n-2)\text{cosec}^{n-2} x\left(\text{cosec}^2 x - 1\right)\mathrm{d}x$	Uses $\cot^2 x = \text{cosec}^2 x - 1$ (incorrect signs allowed)	dM1
	$= -\cot x \,\text{cosec}^{n-2} x - (n-2)\int \text{cosec}^n x \,\mathrm{d}x + (n-2)\int \text{cosec}^{n-2} x \,\mathrm{d}x$		
	$= -\cot x \,\text{cosec}^{n-2} x - (n-2)I_n + (n-2)I_{n-2}$	Introduces I_n and I_{n-2}	ddM1
	$I_n = \dfrac{n-2}{n-1}I_{n-2} - \dfrac{1}{n-1}\cot x \,\text{cosec}^{n-2} x$ *	Correct completion with no errors (apart from possible omission of dx)	A1* cso
			(5)
5(a) Way 2	$I_n = \int \text{cosec}^n x \,\mathrm{d}x = \int \text{cosec}^{n-2} x \,\text{cosec}^2 x \,\mathrm{d}x$		
	$I_n = \int \text{cosec}^{n-2} x \,\text{cosec}^2 x \,\mathrm{d}x = \int \text{cosec}^{n-2} x\left(1 + \cot^2 x\right)\mathrm{d}x$ Uses $\cot^2 x = \text{cosec}^2 x - 1$ (incorrect signs allowed)		M1
	$= \int \text{cosec}^{n-2} x \,\mathrm{d}x + \int \cot^2 x \,\text{cosec}^{n-2} x \,\mathrm{d}x$		
	$\int \cot^2 x \,\text{cosec}^{n-2} x \,\mathrm{d}x = -\dfrac{1}{(n-2)}\cot x \,\text{cosec}^{n-2} x - \dfrac{1}{(n-2)}\int \text{cosec}^n x \,\mathrm{d}x$ M1: Parts in the correct direction A1: Correct expression		dM1 (2nd M on e-PEN) A1 (1st A mark on e-PEN)
	$= I_{n-2} - \dfrac{1}{(n-2)}\cot x \,\text{cosec}^{n-2} x - \dfrac{1}{(n-2)}I_n$	Introduces I_n and I_{n-2}	ddM1
	$I_n = \dfrac{n-2}{n-1}I_{n-2} - \dfrac{1}{n-1}\cot x \,\text{cosec}^{n-2} x$ *	Correct completion with no errors	A1*cso
			(5)

(b)	$I_4 = \frac{2}{3}I_2 - \frac{1}{3}\cot x\,\mathrm{cosec}^2 x$	Correct application of the given reduction formula	M1
	$I_2 = \int \mathrm{cosec}^2 x\,\mathrm{d}x = -\cot x$ or $= -\cot x\,\mathrm{cosec}^0 x$	By integration or use of reduction formula	B1
	$I_4 = \frac{2}{3}(-\cot x) - \frac{1}{3}\cot x\left(1+\cot^2 x\right)$	Uses their I_2 and $\mathrm{cosec}^2 x = 1+\cot^2 x$ (incorrect signs allowed)	M1
	$I_4 = -\cot x - \frac{1}{3}\cot^3 x\,(+c)$	+ c not required	A1cso
			(4)
	(b) Alternative		
	$\int \mathrm{cosec}^4 x\,\mathrm{d}x = \int \left(1+\cot^2 x\right)\mathrm{cosec}^2 x\,\mathrm{d}x$	$\mathrm{cosec}^4 x = \left(1+\cot^2 x\right)\mathrm{cosec}^2 x$	M1
	$= \int \left(\mathrm{cosec}^2 x + \mathrm{cosec}^2 x\cot^2 x\right)\mathrm{d}x$		
	$I_4 = -\cot x - \frac{1}{3}\cot^3 x\,(+c)$	B1: $\int \mathrm{cosec}^2 x\,\mathrm{d}x = -\cot x$ M1: $\int \mathrm{cosec}^2 x\cot^2 x\,\mathrm{d}x = k\cot^3 x$ A1: $\int \mathrm{cosec}^2 x\cot^2 x\,\mathrm{d}x = -\frac{1}{3}\cot^3 x$	B1M1A1cso
			Total 9

Question Number	Scheme	Notes	Marks
6	$\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$		
(a)	$\dfrac{2x}{a^2}-\dfrac{2y}{b^2}\dfrac{dy}{dx}=0\Rightarrow\dfrac{dy}{dx}=\dfrac{b^2}{a^2}\dfrac{a\sec\theta}{b\tan\theta}\left(=\dfrac{b}{a\sin\theta}\right)$ or $\dfrac{dy}{d\theta}=b\sec^2\theta,\dfrac{dx}{d\theta}=a\sec\theta\tan\theta\Rightarrow\dfrac{dy}{dx}=\dfrac{b\sec^2\theta}{a\sec\theta\tan\theta}$ or $y=b\sqrt{\dfrac{x^2}{a^2}-1}\Rightarrow\dfrac{dy}{dx}=\dfrac{b}{2}.\dfrac{2x}{a^2}\left(\dfrac{x^2}{a^2}-1\right)^{-\frac{1}{2}}=\dfrac{ab\sec\theta}{a^2}\left(\sec^2\theta-1\right)^{-\frac{1}{2}}$	M1: Attempts to differentiate implicitly or parametrically or directly to obtain $\dfrac{dy}{dx}=$ A1: Correct derivative as a function of θ Any equivalent form	M1A1
	$y-b\tan\theta=\dfrac{b}{a\sin\theta}\left(x-a\sec\theta\right)$	Correct straight line method Must be complete ie $y=mx+c$ used needs an attempt at c	dM1
	$ay\tan\theta-ab\tan^2\theta=b\sec\theta\left(x-a\sec\theta\right)$		
	$bx\sec\theta-ay\tan\theta=ab$*	Reaches printed answer with at least one intermediate line of working	A1* cso
			(4)
(b)	F is $(ae, 0)$	Correct focus (may be implied but if y coordinate = 0 not used give B0	B1
	$abe\sec\theta=ab\left(\Rightarrow e=\cos\theta\right)$	Substitute the coordinates of the focus into l	M1
	$m=\dfrac{b}{a\sin\theta}=\dfrac{b}{a\sqrt{1-\cos^2\theta}}=\dfrac{b}{a\sqrt{1-e^2}}$	Uses the gradient of l to obtain an expression for m in terms of a, b and e	M1
	For an ellipse $b^2=a^2\left(1-e^2\right)$	Use of the correct eccentricity formula for an ellipse in their expression for m	M1
	So $m=\dfrac{a\sqrt{1-e^2}}{a\sqrt{1-e^2}}=1$ so l is parallel to $y=x$	Correct completion with no errors and conclusion	A1 cso **(5)**
			Total 9

	(b) Way 2		
	$\frac{b}{a\sin\theta}=1 \Rightarrow \sin\theta=\frac{b}{a}$	$\sin\theta=\frac{b}{a}$	B1
	$\sec\theta=\frac{1}{\cos\theta}=\frac{1}{\sqrt{1-\frac{b^2}{a^2}}}=\frac{a}{\sqrt{a^2-b^2}}$	Attempt $\sec\theta$	M1
	$bx\sec\theta-ay\tan\theta=ab \Rightarrow bx\frac{a}{\sqrt{a^2-b^2}}=ab \Rightarrow x=...$	Substitute for $\sec\theta$ and uses $y=0$ and makes x the subject	M1
	$\Rightarrow x=\sqrt{a^2-b^2}$		
	For an ellipse $b^2=a^2\left(1-e^2\right)\Rightarrow ae=\sqrt{a^2-b^2}$	Use of the correct eccentricity formula for an ellipse	M1
	So tangent passes through (ae, 0) which is F	Correct completion with no errors and conclusion	A1 cso
	(b) Way 3		
	Focus is $(ae,0)$		B1
	$=\left(\sqrt{\left(a^2-b^2\right)},0\right)$	Use of the correct eccentricity formula for the ellipse	M1
	Eqn of line: $bx\sec\theta-ay\tan\theta=ab$		
	So $b\sqrt{\left(a^2-b^2\right)}\sec\theta-0=ab$	Line passes thro' the focus	M1
	$\sec\theta=\frac{a}{\sqrt{a^2-b^2}} \Rightarrow \tan\theta=\frac{b}{\sqrt{a^2-b^2}}$	Attempts $\sec\theta$ and $\tan\theta$	M1
	$x-y=\sqrt{a^2-b^2}$ OR Sub $\sec\theta$ and $\tan\theta$ into gradient to get 1	Correct completion with no errors and conclusion	
	$\therefore$ Parallel to $y=x$		A1 cso **(5)**

Question Number	Scheme	Notes	Marks
7	$\int \frac{5+x}{\sqrt{4-3x^2}}\,dx$		
(a)	$\int \frac{5+x}{\sqrt{4-3x^2}}\,dx = \int \frac{5}{\sqrt{4-3x^2}}\,dx + \int \frac{x}{\sqrt{4-3x^2}}\,dx$	Splits (Denominators to be correct) Can be evidenced by the two separate integrals below.	M1
	$\int \frac{5}{\sqrt{4-3x^2}}\,dx = \frac{5}{\sqrt{3}}\arcsin\frac{\sqrt{3}}{2}x$	M1: $p\arcsin qx$ Depends on 1st M mark (p can = 1) A1: $\frac{5}{\sqrt{3}}\arcsin\frac{\sqrt{3}}{2}x$	dM1A1
	$\int \frac{x}{\sqrt{4-3x^2}}\,dx = -\frac{1}{3}\left(4-3x^2\right)^{\frac{1}{2}}$	M1: $k\left(4-3x^2\right)^{\frac{1}{2}}$ Depends on 1st M mark A1: $-\frac{1}{3}\left(4-3x^2\right)^{\frac{1}{2}}$	dM1A1
	$\int \frac{5+x}{\sqrt{4-3x^2}}\,dx = \frac{5}{\sqrt{3}}\arcsin\frac{\sqrt{3}}{2}x - \frac{1}{3}\left(4-3x^2\right)^{\frac{1}{2}}\ (+c)$		
			(5)
	Alternative (a)		
	$x = \frac{2}{\sqrt{3}}\sin u \Rightarrow \int \frac{5+x}{\sqrt{4-3x^2}}\,dx = \int \frac{5+\frac{2}{\sqrt{3}}\sin u}{\sqrt{4-4\sin^2 u}}\frac{2}{\sqrt{3}}\cos u\,du$ M1: Attempt $x = k\sin u$ including replacing dx		M1
	$= \frac{5}{\sqrt{3}}u - \frac{2}{3}\cos u\ (+c)$	M1: ku or $k\cos u$ Depends on 1st M A1: Both correct	dM1A1
	$= \frac{5}{\sqrt{3}}\arcsin\frac{\sqrt{3}}{2}x - \frac{1}{3}\left(4-3x^2\right)^{\frac{1}{2}}\ (+c)$ or $\frac{5}{\sqrt{3}}\arcsin\frac{\sqrt{3}}{2}x - \frac{2}{3}\cos\left[\arcsin\left(\frac{\sqrt{3}}{2}x\right)\right]$	M1: Changes back to x Depends on both preceeding M marks A1: Fully correct (Allow equivalent correct forms)	ddM1A1
	Can be done by sub $x = \frac{2}{\sqrt{3}}\tanh\theta$.		
(b)	$\left[\frac{5}{\sqrt{3}}\arcsin\frac{\sqrt{3}}{2}x - \frac{1}{3}\left(4-3x^2\right)^{\frac{1}{2}}\right]_0^1$		
	$\left[\frac{5}{\sqrt{3}}\arcsin\frac{\sqrt{3}}{2} - \frac{1}{3}\right] - \left[\frac{5}{\sqrt{3}}\arcsin 0 - \frac{1}{3}\times 2\right]$	Substitute the limits 0 and 1 (or 0 and $\frac{\pi}{3}$ if in terms of u) in **both** parts of their integral from (a) and subtract the right way round.	M1
	$= \frac{5}{9}\pi\sqrt{3}, +\frac{1}{3}$	Any exact equivalent	A1, A1
			(3)
			Total 8

Question Number	Scheme	Notes	Marks
8	$x=\theta-\sin\theta,\quad y=1-\cos\theta,\qquad 0\le\theta\le 2\pi$		
(a)	$\dfrac{dx}{d\theta}=1-\cos\theta,\qquad \dfrac{dy}{d\theta}=\sin\theta$	Both	B1
	$(S=2\pi)\int(1-\cos\theta)\sqrt{1-2\cos\theta+\cos^2\theta+\sin^2\theta}\ d\theta$ or $\int(1-\cos\theta)\sqrt{(1-\cos\theta)^2+\sin^2\theta}\ d\theta$	M1: Uses the correct formula with their derivatives. Integrand must be a function of θ A1: Correct integrand	M1A1
	$(S=2\pi)\int(1-\cos\theta)\sqrt{2(1-\cos\theta)}\ d\theta$		
	$S=2\pi\sqrt{2}\int_0^{2\pi}(1-\cos\theta)^{\frac{3}{2}}d\theta$*	cso with at least one intermediate step shown	A1*cso
			(4)
(b)	$S=2\pi\sqrt{2}\int\left(1-\left(1-2\sin^2\dfrac{\theta}{2}\right)\right)^{\frac{3}{2}}d\theta$	Uses $\cos\theta=\pm1\pm2\sin^2\dfrac{\theta}{2}$	M1
	$=8\pi\int\sin^3\dfrac{\theta}{2}d\theta$	Correct expression	A1
	$=8\pi\int\sin\dfrac{\theta}{2}\left(1-\cos^2\dfrac{\theta}{2}\right)d\theta$	Uses Pythagoras	dM1
	$=8\pi\left[-2\cos\dfrac{\theta}{2}+\dfrac{2}{3}\cos^3\dfrac{\theta}{2}\right]_0^{2\pi}$	Integrates to obtain $p\cos\dfrac{\theta}{2}+q\cos^3\dfrac{\theta}{2},\ \ p,q\ne0$	ddM1
	$=16\pi\left[\left(-\dfrac{1}{3}+1\right)-\left(\dfrac{1}{3}-1\right)\right]$	Include 16π and use limits correctly	dddM1
	$=\dfrac{64\pi}{3}$		A1 **(6)**
			Total 10
	Alternatives for (b)		
1	$S=2\pi\sqrt{2}\int\left(1-\left(1-2\sin^2\dfrac{\theta}{2}\right)\right)^{\frac{3}{2}}d\theta$	Uses $\cos\theta=\pm1\pm2\sin^2\dfrac{\theta}{2}$	M1
	$=8\pi\int\sin^3\dfrac{\theta}{2}d\theta$	Correct expression	A1
	$\sin^3\dfrac{\theta}{2}=\dfrac{3}{4}\sin\dfrac{\theta}{2}-\dfrac{1}{4}\sin\dfrac{3\theta}{2}$		
	$8\pi\int\left(\dfrac{3}{4}\sin\dfrac{\theta}{2}-\dfrac{1}{4}\sin\dfrac{3\theta}{2}\right)d\theta$	Uses the above identity (sign errors allowed)	dM1
	$=8\pi\left(-\dfrac{3}{4}\times2\cos\dfrac{\theta}{2}+\dfrac{1}{4}\times\dfrac{2}{3}\cos\dfrac{3\theta}{2}\right)$	Integrates to obtain $p\cos\dfrac{\theta}{2}+q\cos\dfrac{3\theta}{2}$	ddM1
	$8\pi\left[\left(\dfrac{3}{2}-\dfrac{1}{6}\right)-\left(\dfrac{3}{2}+\dfrac{1}{6}\right)\right]=\dfrac{64\pi}{3}$	Correct use of limits	dddM1 A1

2.	$u=1-\cos\theta;\ \mathrm{d}u=\sin\theta\mathrm{d}\theta$		
	$\sin\theta=\sqrt{1-\cos^2\theta}=\sqrt{1-(1-u)^2}$ $=u^{\frac{1}{2}}(2-u)^{\frac{1}{2}}$		
	$\int(1-\cos\theta)^{\frac{3}{2}}\mathrm{d}\theta=\int u(2-u)^{-\frac{1}{2}}\mathrm{d}u$	M1 Attempt the substitution. Integral to be in terms of u only. A1 Correct integral in terms of u	M1A1
	$=-2u(2-u)^{\frac{1}{2}}-\int -2(2-u)^{\frac{1}{2}}\times 1\mathrm{d}u$		
	$=-2u(2-u)^{\frac{1}{2}}-2\times\frac{2}{3}(2-u)^{\frac{3}{2}}$	Integrate by parts	dM1
	$2\pi\sqrt{2}\int_0^{2\pi}(1-\cos\theta)^{\frac{3}{2}}\mathrm{d}\theta$ $=2\times 2\pi\sqrt{2}\int_0^{\pi}(1-\cos\theta)^{\frac{3}{2}}\mathrm{d}\theta$		
	$2\times 2\pi\sqrt{2}\int_0^{2}u(2-u)^{-\frac{1}{2}}\mathrm{d}u$ $=2\times 2\pi\sqrt{2}\times 2\times\frac{2}{3}\times 2^{\frac{3}{2}}$	Include the constant multiplier of the integral and EITHER: M1 Change the limits M1 Substitute limits for u OR: M1 Reverse the substitution M1 Substitute limits for θ	ddM1 ddM1
	$.=\frac{64\pi}{3}.$		A1
3	**Using Integration by parts:**		
	$S=8\pi\int\sin^3\frac{\theta}{2}\mathrm{d}\theta$	See main scheme	M1A1
	$u=\sin^2\frac{\theta}{2}$ $\quad\frac{\mathrm{d}v}{\mathrm{d}\theta}=\sin\frac{\theta}{2}$ $\frac{\mathrm{d}u}{\mathrm{d}\theta}=\sin\frac{\theta}{2}\cos\frac{\theta}{2}$ $\quad v=-2\cos\frac{\theta}{2}$		
	$=-2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}+\int 2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}\mathrm{d}\theta$	Integrate by parts	dM1
	$\int 2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}\mathrm{d}\theta\to k\cos^3\frac{\theta}{2}$		ddM1
	$=\frac{64\pi}{3}$	M1: Include the constant multiplier and use limits (0 and 2pi) correctly A1: Correct answer	dddM1A1

Question Number	Scheme	Notes	Marks
9	$A(-1, 5, 1),\ B(1, 0, 3),\ C(2, -1, 2),\ D(3, 6, -1)$		
(a)	$\mathbf{AB} = \begin{pmatrix}2\\-5\\2\end{pmatrix},\ \mathbf{AC} = \begin{pmatrix}3\\-6\\1\end{pmatrix},\ \mathbf{AD} = \begin{pmatrix}4\\1\\-2\end{pmatrix}$ $\mathbf{DB} = \begin{pmatrix}2\\6\\-4\end{pmatrix},\ \mathbf{DC} = \begin{pmatrix}1\\7\\-3\end{pmatrix},\ \mathbf{BC} = \begin{pmatrix}1\\-1\\-1\end{pmatrix}$	Attempts 3 edges of the tetrahedron Any triple with a common vertex Method to be shown or at least 1 correct	M1
	$\begin{vmatrix}2 & -5 & 2\\3 & -6 & 1\\4 & 1 & -2\end{vmatrix}$ or $\begin{pmatrix}2\\-5\\2\end{pmatrix}\cdot\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\3 & -6 & 1\\4 & 1 & -2\end{vmatrix}$	Attempt appropriate triple product with their edges. (M0 if a vector is obtained)	dM1
	$= \frac{1}{6}(22-50+54) = \frac{13}{3}\ \left(4\frac{1}{3} \text{ or } 4.3\text{rec}\right)$	dM1: Completes including $\frac{1}{6}$ (depends on both M marks above) A1: Correct volume (allow equivalents)	ddM1A1
			(4)
	Cartesian method: Find the equation of a plane containing a face of the tetrahedron		M1
	Then find area of triangle and perp height		dM1
	Complete by using Vol $= \frac{1}{3} \times \text{base area} \times \text{height} = \frac{13}{3}$		ddM1A1 (4)
(b)	$\mathbf{AB} \times \mathbf{AC} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\2 & -5 & 2\\3 & -6 & 1\end{vmatrix} = \begin{pmatrix}7\\4\\3\end{pmatrix}$	M1: Attempt cross product between two sides of *ABC* Min one element correct. A1: Correct normal vector (any multiple)	M1A1
	$\begin{pmatrix}7\\4\\3\end{pmatrix}\cdot\begin{pmatrix}-1\\5\\1\end{pmatrix}(=16)$	Attempt scalar product using their normal vector Answer correct for their vectors or method shown.	dM1
	$7x+4y+3z=16$	Correct equation (any multiple)	A1 **(4)**
	Alternative to (b)		
	$-a+5b+c=d,\ a+3c=d,\ 2a-b+2c=d$	Uses A, B and C to form 3 equations	M1
	$a=7,\ b=4,\ c=3$	Correct values	A1
	$\begin{pmatrix}7\\4\\3\end{pmatrix}\cdot\begin{pmatrix}-1\\5\\1\end{pmatrix}(=16)$		dM1
	$7x+4y+3z=16$	Correct equation (any multiple)	A1

(c)	$\mathbf{DT}=\begin{pmatrix}3\\6\\-1\end{pmatrix}+\lambda\begin{pmatrix}7\\4\\3\end{pmatrix}$	Attempt parametric form of **DT** using their normal vector	M1
	$7(3+7\lambda)+4(6+4\lambda)+3(-1+3\lambda)=16$ $\Rightarrow \lambda=...$	Substitutes parametric form of **DT** into their plane equation and solves for λ	dM1
	$\lambda=-\dfrac{13}{37}\Rightarrow T$ is $\left(\dfrac{20}{37},\dfrac{170}{37},-\dfrac{76}{37}\right)$	M1: Uses their value of λ in their **DT** equation. Can be indicated by any coordinate correct for their **DT** and λ A1: Correct exact coordinates Or correct vector $\overrightarrow{OT}$	ddM1A1
			(4)
			Total 12